MATH 2301 — 29 Aug 2022

* Matrix representation for $\mathcal{A}(P)$

If $f \in \mathcal{A}(P)$, (choose ordering $(x_1, \dots x_n)$ of P)

$$M_f = \begin{array}{c} \\ x_i \end{array}\left[\begin{array}{ccc} & x_j & \\ & \vdots & \\ \cdots & \boxed{} & \cdots \\ & & \end{array}\right] \quad \boxed{} = \begin{cases} f([x_i,x_j]) & \text{if } x_i \leq x_j \\ 0 & \text{otherwise.} \end{cases}$$

* $M_{(f*g)} = M_f \cdot M_g$
* $M_{f+g} = M_f + M_g$
* $M_{f^{-1}} = (M_f)^{-1}$

[Recall that $I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ = identity matrix

& $M_\delta = I$.]

We say that a square matrix A is invertible with inverse $B = A^{-1}$ if:

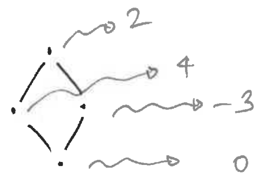
$$A \cdot B = B \cdot A = I$$

* Rmk: We don't know (in this class) techniques to compute matrix inverses → Linear Algebra.

* One-sided convolution of functions on posets

Def: A function on a poset $(P, \preceq)$ is just any function $p : P \to \mathbb{R}$.

E.g.



example function

One way to write down such a thing is via a vector:

$$\begin{bmatrix} p(x_1) \\ p(x_2) \\ \vdots \\ p(x_n) \end{bmatrix} = N_p$$

$(n \times 1)$ matrix / column vector

If $(x_1, x_2, \dots, x_n)$ are the elements of the poset P

Call this N_p

Def: Let $f \in \mathcal{A}(P)$ & $p : P \to \mathbb{R}$ be a function on P.

Then the one-sided convolutions are:

(1) $(f*p) : P \to \mathbb{R}$, defined as:

$$(f*p)(x) = \sum_{x \preceq z} \underbrace{f([x,z])}_{\text{real number}} \cdot \underbrace{p(z)}_{\text{real number}}$$

(2) $(p*f)(x) = \sum_{z \preceq x} p(z) \cdot f([z,x])$.

Prop: If $f \in \mathcal{A}(P)$, $p: P \to \mathbb{R}$ is a function, let $\underset{(n\times n)}{M_f}$ and $\underset{(n\times 1)}{v_p}$ be the associated matrices.

Let v_p^t be the transpose of v_p.

$\left(v_p^t = [p(x_1),\ p(x_2),\ \dots\ p(x_n)] \right)$

(1) $M_f \cdot v_p = v_{(f*p)}$ ← the vector of the one-sided convolution is the product of the matrix of f & the vector of p; on the appropriate sides

(2) $v_p^t \cdot M_f = v^t_{(p*f)}$

[Examples later ...]

* What we're moving towards: Möbius inversion

(A general version of principle of inclusion & exclusion → a counting technique.)

$\mu \in \mathcal{A}(P)$ is the Möbius function, such that

$(\zeta * \mu) = (\mu * \zeta) = \delta$.

Eg.  [for concreteness, if you like...]

. Let $(P, \preceq)$ be any poset

Let us find a formula for μ.

① For intervals of type $[x,x]$:

$(\mu * \zeta)([x,x]) = \mu([x,x]) \cdot \zeta([x,x])$ [summation simplifies]

$= \delta([x,x]) = 1$

$\Rightarrow \forall\, x \in P,\ \mu([x,x]) = 1$.

② Let $x, y \in P$ such that $x \preceq y$ & $x \neq y$

$0 = \delta([x,y]) = (\mu * \zeta)([x,y])$

$= \sum_{x \preceq z \preceq y} \mu([x,z])\ \overbrace{\zeta([z,y])}^{=1}$

$$\boxed{0 = \sum_{x \preceq z \preceq y} \mu([x,z])}$$

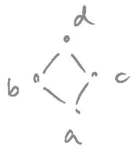
$$0 = \mu([x,y]) + \sum_{x \preceq z \precneqq y} \mu([x,z])$$

$\Longrightarrow$ Recursive formula for μ:

$$\boxed{\mu([x,y]) = -\sum_{x \preceq z \precneqq y} \mu([x,z])}$$

Use this to bootstrap values of μ from the fact that $\mu([x,x]) = 1$.

E.g.



Next : compute $\mu([a,b])$

$\mu([a,c])$, etc,

finally $\mu([a,d])$