MATH 2301 — 31 Aug 2022

* Last time

$$\boxed{\mu([x,y]) = -\sum_{x \le z \lneq y} \mu([x,z])}$$

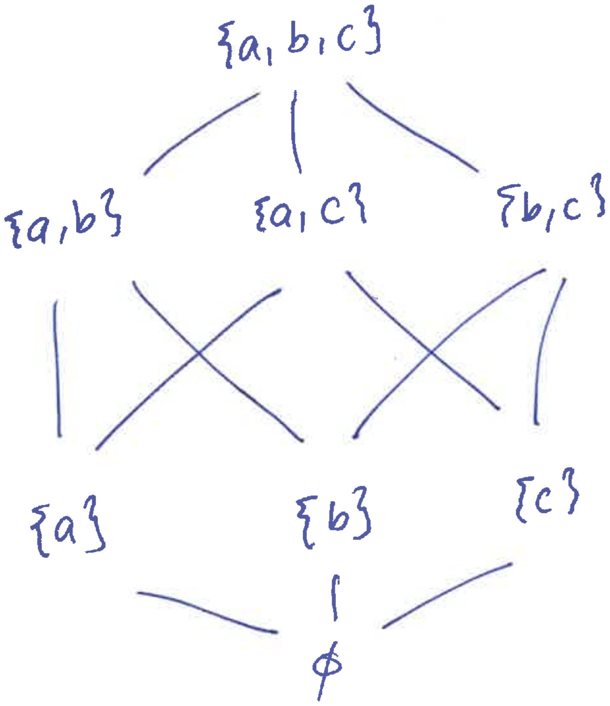
if $x < y$, and $\mu([x,x]) = 1$

* Today: Calculations using this.

Focus on two examples:

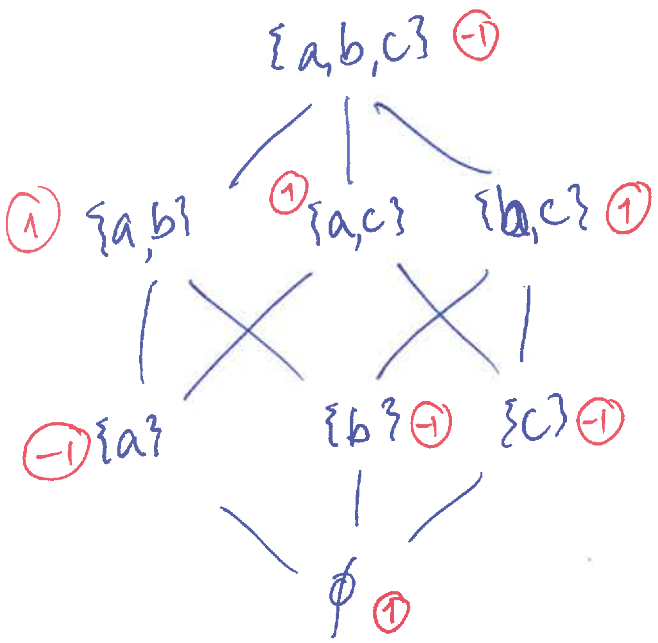
1. the subset poset of S = {set of all subsets of S} $= \mathcal{P}(S)$ with $\subseteq$ as the partial order
2. the divisor poset of n = set of all divisors of n with "$|$" = divisibility as the relation.

* Example: the subset poset of $\{a,b,c\}$



Try to compute $\mu([\phi, \{a,b,c\}])$

(shorthand drawing)



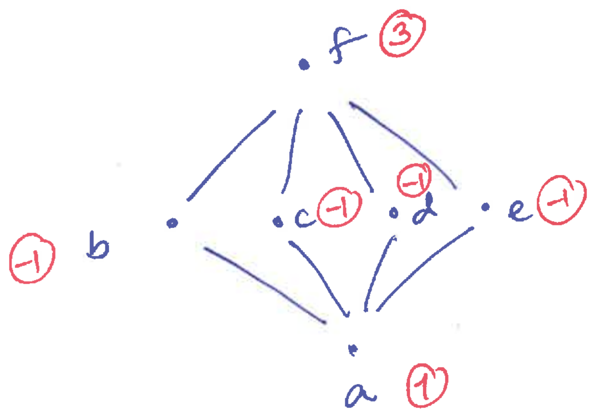
Numbers in red are $\mu([\phi, X])$ for each X.

$$\mu([\phi,\{a\}]) = -\sum_{\phi \le z \lneq \{a\}} \mu([\phi, z])$$

$$= -\mu([\phi,\phi]) = -1$$

$$\mu([\phi, \{b,c\}]) = -\mu([\phi,\phi]) - \mu([\phi,\{b\}]) - \mu([\phi,\{c\}])$$

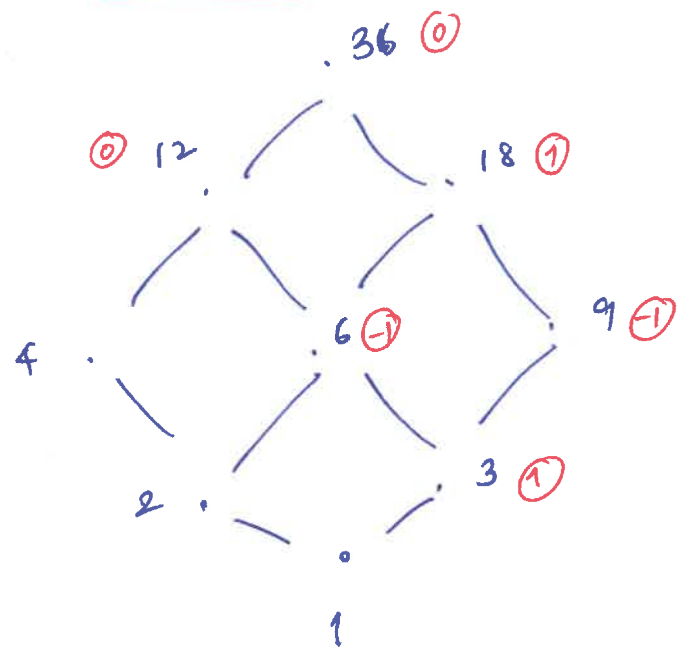
* Example



$\mu([a,f])$

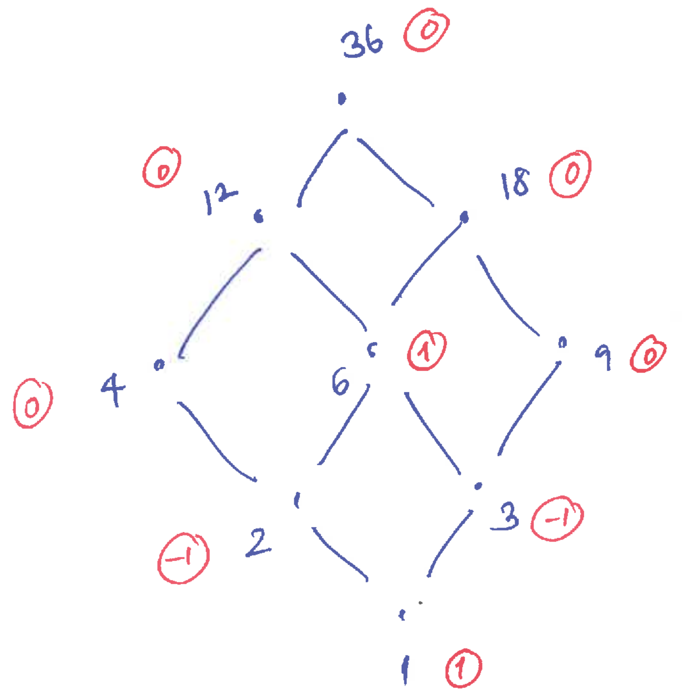
(Numbers in red are $\mu([a,x])$)

* **Example**: Divisor poset of 36.



$\mu([3,36])$?

Numbers in red indicate $\mu([3,x])$



$\mu([1,36])$

Numbers in red are $\mu([1,x])$

* **Thm**: Let S be a set and consider its subset poset. Then, if A, B are subsets of S, then:

① $\mu([\emptyset, A]) = (-1)^{|A|}$ ← size of A

② If $A \subseteq B$, then
$$\mu([A, B]) = (-1)^{|B \setminus A|}$$
($B \setminus A$ = all elements of B that are not in A.)

[Pf skipped]

* **Back** to ~~subset~~ divisor **poset**

Example: Let p be a prime number. Look at divisor poset of p^n for some n.

0	p^n
⋮	⋮
0	p^3
0	p^2
−1	p
1	1

Only divisors of p^n are p^k. for $k \le n$.

$$\mu([1, p^n]) = \begin{cases} 1 & \text{if } n = 0 \\ -1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

* <u>Thm</u>: Let m be a positive integer.

Write the prime power decomposition of m:

$$m = p_1^{n_1} \, p_2^{n_2} \cdots p_k^{n_k}$$ [each p_i is a different prime]

$\mu([1, m])$ in the divisor poset is:

$$\mu([1,m]) = \mu([1, p_1^{n_1}]) \cdot \mu([1, p_2^{n_2}]) \cdot \cdots \cdot \mu([1, p_k^{n_k}])$$

<u>Example</u>: $m = 36 = 2^2 \cdot 3^2$

$$\mu([1,36]) = \mu([1, 2^2]) \cdot \mu([1, 3^2])$$

$$= 0 \cdot 0 = 0$$

(<u>Pf skipped</u>)