MATH 2301 — 1 . Sep 2022

* Last time

① Subset poset of some S

$$\mu([A,B]) = (-1)^{|B \setminus A|}$$

② Divisor poset of some $m \geq 1$.

(a) If $m = p^n$ for some prime p, then

$$\mu([1,p^n]) = \begin{cases} 1 & \text{if } n=0 \\ -1 & \text{if } n=1 \\ 0 & \text{if } n>1 \end{cases}$$

(b) If ~~$m = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$~~ ~~for~~

$m = p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$ for distinct primes $p_1, p_2, \dots, p_r$

then

$$\mu([1,m]) = \mu([1,p_1^{n_1}]) \cdot \mu([1,p_2^{n_2}]) \cdot \cdots \cdot \mu([1,p_r^{n_r}])$$

Example : $m = 24 = 3^1 \cdot 2^3$

$$\mu([1,24]) = \underbrace{\mu([1,3])}_{=-1} \cdot \underbrace{\mu([1,8])}_{=0} = 0.$$

in the divisor poset

Rmk : $\mu([1,m]) = 0$ if m is divisible by the square of any prime.

Otherwise, m is the product of distinct primes, and $\mu([1,m]) = (-1)^{(\text{number of primes})}$.

(c) Thm: If $a \mid b$ then

$$\mu([a,b]) = \mu([1, b/a])$$

E.g. $\mu([4,24]) = \mu([1,6])$

$$= \mu([1,2]) \cdot \mu([1,3])$$

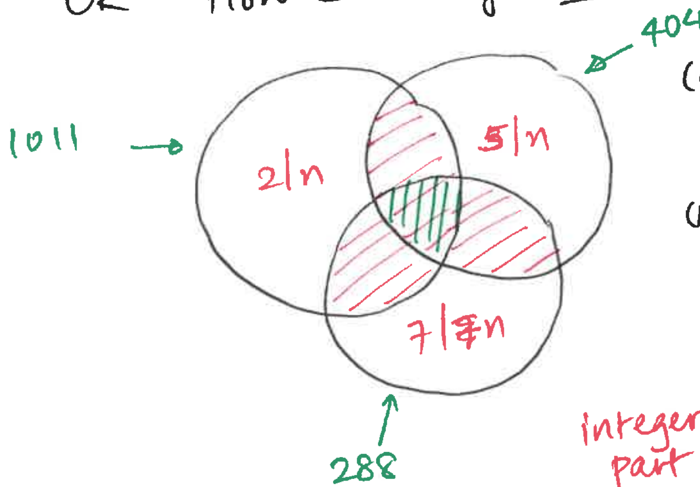
$$\mu([4,24]) = (-1)\cdot(-1) = 1.$$

---

* Möbius inversion or principle of inclusion/exclusion (PIE)

Example question: How many positive natural numbers between 1 and 2022 are not divisible by any of the following: 2, 5, 7?

OR: How many are divisible by one of the three?



(a) How many are divisible by 2?

$\rightarrow$ 1011

(b) Divisible by 5?

$$\rightarrow \left\lfloor \frac{2022}{5} \right\rfloor = 404$$

integer part

(c) Divisible by 7?

$$\rightarrow \left\lfloor \frac{2022}{7} \right\rfloor = 288.$$

Want to add them, then remove overlaps

(d) Divisible by 2 and 5 ?
i.e. divisible by 10?

$\to \left\lfloor \frac{2022}{10} \right\rfloor = 202$

(e) 2 and 7 ? $\to \left\lfloor \frac{2022}{14} \right\rfloor = \left\lfloor \frac{1011}{7} \right\rfloor = 144$.

(f) 5 and 7 ? $\to \left\lfloor \frac{2022}{35} \right\rfloor = 57$ (?)

(g) 2, 5, and 7 ? $\to \left\lfloor \frac{2022}{70} \right\rfloor = 28$ (?)

# of integers ~~&~~ from 1 to 2022 that are divisible by at least one of 2, 5, 7 is:

$$\underbrace{(1011 + 404 + 288)}_{\text{(singles)}} - \underbrace{(202 + 144 + 57)}_{\text{(pairs)}} + \underbrace{28}_{\text{(triple)}}$$

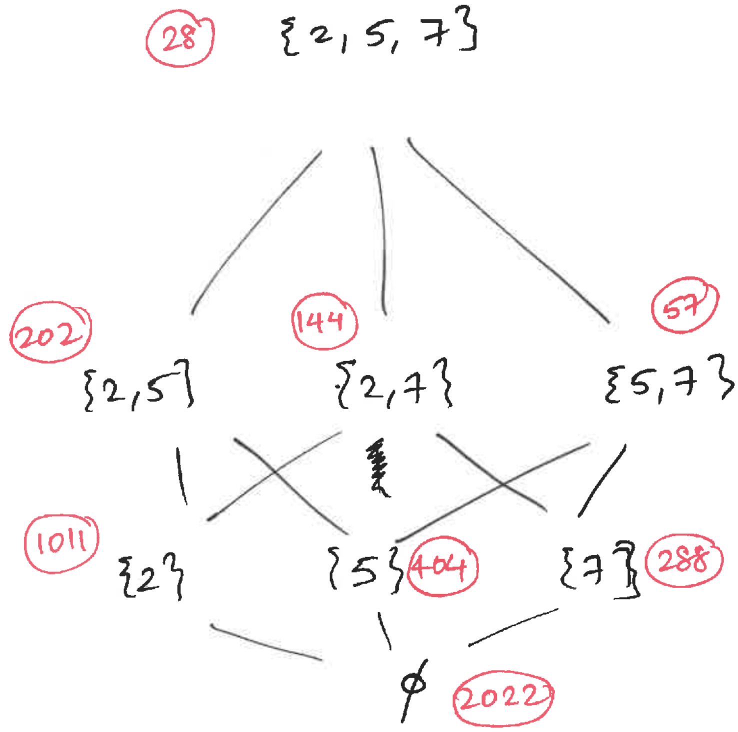
$$\stackrel{?}{=} 1328$$

~~of~~ divisible by none of them:

$(2022 - \cancel{1353}\ 1328) = \boxed{694}$.

Plan : rephrase in terms of incidence algebra.

Subset poset of $\{2,5,7\}$



Had a function

$p : P \to \mathbb{R}$

$p(A) =$ # integers from 1 to 2022 that are divisible by all elements of A

(we'll write the values in red)

Let's define

$q : P \to \mathbb{R}$

$q(A) =$ # integers from 1 to 2022 that are divisible by exactly the elements of A, and no others.

E.g. $q(\{2,5\}) = (202 - 28)$

$\Rightarrow$ # integers from 1 to 2022 divisible by none of the 3 numbers is precisely $q(\phi)$.

* Magic from posets

Observe: $p(A) = \sum_{A \subseteq B} q(B) = (\zeta * q)(A)$

i.e $\boxed{p = (\zeta * q)}$.

Multiply on the left by μ:

$$\mu * p = \mu * (\zeta * q) = (\mu * \zeta) * q$$

$$\mu * p = \delta * q = q$$

$$\boxed{\mu * p = q}$$ Möbius inversion formula.

$$\Rightarrow \quad q(\phi) = \sum_{\phi \subseteq X} \mu([\phi, X]) \cdot p(X)$$

$$q(\phi) = \sum_{\phi \subseteq X} \cancel{\mu}\,(-1)^{|X|} \cdot p(X)$$