MATH 2301 12 Oct 2022

* Last time : Regular & non-regular languages

[Recall : A language L is regular if one of the following, equivalently all of them, hold:

(1) $L = L(r)$ for a regex r

(2) $L = L(M)$ for an NFA M

(3) $L = L(M')$ for a DFA M'.]

* Example : $L = \{0^n 1^n \mid n \geq 0\}$ is not regular

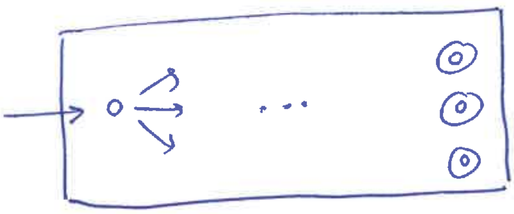
* Today : The pumping lemma.

[A tool to sometimes detect whether a language is not regular.]

Key observation : A feature common to all regular languages

Let L be a regular language $\Rightarrow$ there is a DFA M, such that $L(M) = L$.

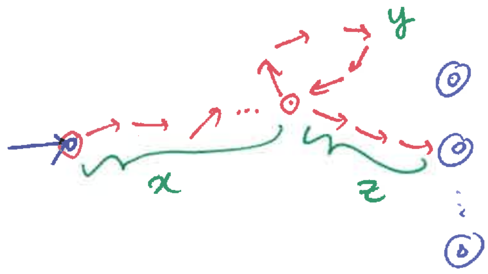
Let n be the number of states in M.



Let w be a word in L

If w is long enough, say $|w| > n$, then there must be a "non-trivial loop".

i.e, there must be a state through which w passes twice. [Compare with paths in graphs.]



This means :

$w = x \underbrace{y}z$
looped portion.

& $y \neq \varepsilon$.

$\Rightarrow xz \in L$, and $xyz, xyyz, xyyyz, \ldots$
[xy^kz $\forall k$] are all in L.

Process of taking xyz & looking at
$xz, xyz, xyyz, xyyyz, \ldots$ [i.e xy^*z]
is known as "pumping" w at y.

Prop: If L is regular, then there is some n such that every string $w \in L$ with $|w| > n$ can be "pumped" to strings that all belong to L.

Example:

$w = 0011 \in \{0^n 1^n \mid n \geq 0\}$

Note: w is not pumpable, ie, there is no decomposition $w = xyz$, such that $xy^kz \in L$ for all k.

Theorem (Pumping Lemma) : Let L be a regular language. Then there exists some $n = n_L \in \mathbb{N}$, such that if $w \in L$, with $|w| > n$, then:

"pumping length for L"

$w = xyz$ such that:

(1) $|y| \geq 1$, i.e. $y \neq \varepsilon$

(2) $xy^kz \in L$ for every $k \geq 0$

(3) $|xy| < n$. ← useful for calculations.

Example : To show that $L = \{0^k 1^k \mid k \geq 0\}$ is not regular.

We'll prove by contradiction.

Suppose L is ~~were~~ regular. Then there is a pumping length n for L. & the pumping lemma holds.

Consider the string $w = 0^n 1^n$; note $|w| > n$.

$\Rightarrow w = xyz$ with $|xy| < n$, $|y| \geq 1$ and $xy^kz \in L \ \forall \ k \geq 0$.

Since $|xy| < n$, we see that $x = 0^{\ell}$ for some $\ell \geq 0$, $y = 0^m$ for some $m \geq 1$.

So $z = 0^{n-m-\ell} 1^n$.

Look at, e.g. the string $xz = 0^{\ell} 0^{n-m-\ell} 1^n$
$= 0^{n-m} 1^n$

The pumping lemma says that $xz = 0^{n-m} 1^n \in L$.

But xz is not of the form $0^k 1^k$ → Contradiction!

Conclusion: Our assumption was wrong!

$\Rightarrow$ L cannot be regular.

---

* Games (Combinatorial game theory)

A game is a model of interaction between two (or more) players, via a prescribed set of "moves".