

\* Games [Impartial combinatorial games]

An impartial combinatorial game is:

- played between two players;
- turn-based (players take turns).
- There is a game state before every turn, and the possible moves depend only on this state (not on who goes next).
- It is perfect information (no secrets)
- It is deterministic (no dice rolls / luck)
- The game ends if the next player has no move to make.
- There are finitely many possible game states after the first state  $\Rightarrow$  each game stops: (reachable)
- No backtracking, i.e. no directed cycles of moves between game states.

\* Example: Subtraction game

Fix some finite set  $S = \{1, 3, 4\}$

Start state = some  $n \geq 1$

Move = ~~at~~ subtract ~~some~~ exactly one  $k \in S$  from  $n$ , to produce  $(n-k) \geq 0$

Let's start with  $n=10$ ,  $S = \{1, 3, 4\}$

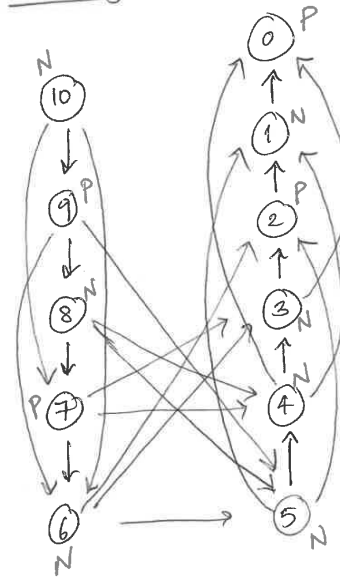
$$10 \rightarrow 9 \rightarrow 6 \rightarrow 3 \rightarrow 0$$

$$25 \rightarrow 21 \rightarrow 18 \rightarrow 17 \rightarrow 16 \rightarrow 13 \rightarrow 9 \rightarrow 8$$

$$0 \leftarrow \frac{1}{2} \leftarrow \frac{2}{4} \leftarrow 6 \leftarrow 7$$

Goal: Who wins?? Which player (1 or 2) has a winning strategy for a given start state?

\* Use game graph



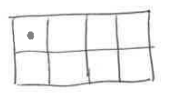
Label each position (state) as "N" or "P"  
 ↓ winning for the next player  
 ↓ win for previous player

- Terminal positions (those with no outgoing arrows) are P
- Anything with an arrow to a P-position is an N
- If a state only has edges to N-positions, then it is a P-state.

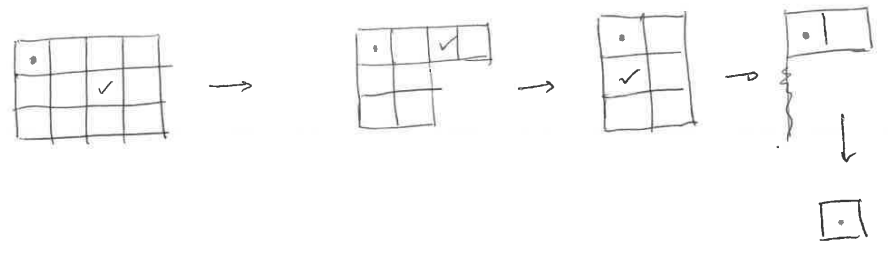
How to win if you are at an N-state?  
 $\rightarrow$  Move to a P-state.

\* Chomp

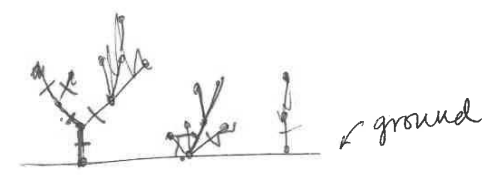
An  $m \times n$  bar of chocolate  
Top left square is poisoned & inedible.



A move: pick a non-poisoned square, and eat everything in the rectangle to the right & below (eat all squares below + right)  
The person who cannot make a move loses.



\* Hackenbush



Game state = forest as shown.

Move = chop off a single branch. Anything not connected to the ground falls off.

\* Look up: Kayles, Sprouts, Wythoff's game, ...  
(on Wikipedia)