

\* Today: Nim

Game state = some number of piles/heap<sup>s</sup>, <sup>each</sup> consisting of some number ( $\geq 0$ ) of berries.

(order is irrelevant, but keep track of multiplicity)

A move consists of eating a non-zero number of berries from exactly one pile.

E.g.:  $(1, 2, 2) \neq (2, 2, 1)$  (game states are tuples) ~~not~~ ~~reordering~~

$\begin{matrix} \swarrow & \downarrow & \searrow & \rightarrow \\ (0, 2, 2) & (1, 1, 2) & (1, 0, 2) & \dots \end{matrix}$

The game ends when there are no moves.

\* Easy cases

- If  $\exists$  exactly one pile of <sup>a non-zero # of</sup> berries, then the first player wins.

$\Rightarrow$  Any position of the form  $(m)$ , for ~~some~~  $m > 0$ , is an N-position.

-  $\exists$  Positions of the form  $(a, b)$  for  $a, b > 0$

$(N) (5, 8)$  or  $(4, 4) (P)$   
 $\downarrow$   
 $(2, 4) \rightarrow (2, 2)$   
 $\downarrow$   
 $\dots$

\* Claim: A position of the form  $(a, b)$  where  $a, b \geq 0$  and  $a \neq b$  is an N-position

A position of the form  $(a, b)$  with  $a = b$  is a P-position.

Pf (sketch): If  $(a, b)$  is such that ~~where~~  $a \neq b$  and  $a, b \geq 0$ , then, suppose (WLOG) that  $a > b$

The winning move is

$$(a, b) \mapsto (b, b)$$

Subsequently, no matter what the second player does, the first player can "mirror" the move.

E.g.  $(4, 7) \rightarrow (4, 4) \rightarrow (1, 4) \rightarrow (1, 1)$

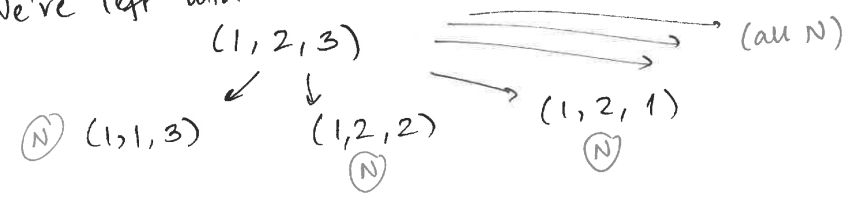
Similarly, from a position of the form  $(a, a)$ , the only possible moves go to positions of the form  $(c, d)$  with  $c \neq d$ .

\* Lesson we learn: "Mirroring" is a good strategy to keep in mind.

Q: What about starting states with  $> 2$  non-empty piles?

E.g. (1, 2, 3)

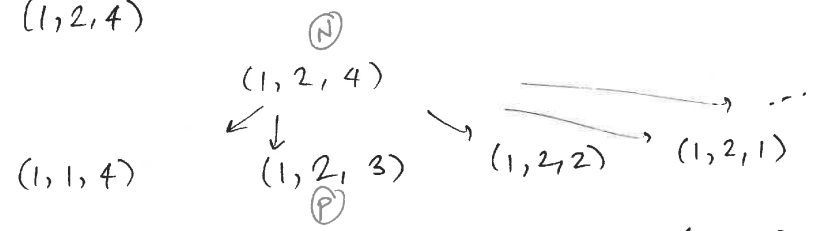
Observe: It's not in P1's favour to eat any single pile  
We've left with:



⇒ (1, 2, 3) is P.

(Note: (1, 1, 3) is N because there is a move to (1, 1, 0), which is P)

E.g. (1, 2, 4)



(1, 2, 4) is N because it has a move to (1, 2, 3), which is P.

Let's work towards a solution

Given a position  $(m_1, m_2, \dots, m_k)$  we're going to compute its "nim-sum":  $m_1 \oplus m_2 \oplus \dots \oplus m_k$ .

= decimal corresponding to the XOR of the binary representation of  $m_1, m_2, m_3, \dots, m_k$ .

The nim-sum is defined in terms of XOR = exclusive or.

Given some  $m$ , we can convert it to binary.

E.g.  $m = 5 = 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 2^2 + 1$

$5 = (101)_2$

Expand  $m$  as a sum of distinct powers of 2, something like  $a_k \cdot 2^k + a_{k-1} \cdot 2^{k-1} + a_{k-2} \cdot 2^{k-2} + \dots + a_0 \cdot 2^0$  where each  $a_i$  is either 1 or 0.

Then  $m = (a_k a_{k-1} \dots a_0)_2$

E.g.  $19 = 16 + 3 = 16 + 2 + 1 = 1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2 + 1 \cdot 2^0$

$19 = (10011)_2$

E.g. (nim sum) of (5, 7, 19)

$$\begin{array}{r}
 5 = \quad \quad 101_2 \\
 \oplus 7 = \quad \quad 111_2 \\
 \oplus 19 = \oplus 10011_2 \\
 \hline
 17 \quad \quad 10001_2
 \end{array}$$

(column-wise XOR)

$$\begin{bmatrix}
 1 \oplus 1 = 0 \\
 1 \oplus 0 = 1 \\
 0 \oplus 1 = 1 \\
 0 \oplus 0 = 0
 \end{bmatrix}$$