MATH 2301 — 17 Oct 2022

* Today : Nim

Game state = some number of piles/heaps each consisting of some number (≥ 0) of berries.
(order is irrelevant, but keep track of multiplicity)

A move consists of eating a non-zero number of berries from exactly one pile.

E.g. : $(1,2,2)$ (game states are tuples) ~~up to reordering~~

$(1,2,2) \to (0,2,2)$, $(1,1,2)$, $(1,0,2)$, …

The game ends when there are no moves.

* Easy cases

- If $\exists$ exactly one pile of a non-zero # of berries, then the first player wins.
  $\Rightarrow$ Any position of the form (m), for $m>0$, is an N-position.

- Positions of the form (a,b) for $a,b>0$

  Ⓝ $(5,8)$ or $(4,4)$ Ⓟ

  $(4,4) \to (2,4) \to (2,2) \to \ldots$

* Claim : A position of the form (a,b) where $a,b \geq 0$ and $a \neq b$ is an N-position.

A position of the form (a,b) with $a=b$ is a P-position.

Pf (sketch) : If (a,b) is such that $a \neq b$ and $a,b \geq 0$, then, suppose (WLOG) that $a>b$.
The winning move is
$(a,b) \longmapsto (b,b)$
Subsequently, no matter what the second player does, the first player can "mirror" the move.
E.g. $(4,7) \to (4,4) \to (1,4) \to (1,1)$

Similarly, from a position of the form (a,a), the only possible moves go to positions of the form (c,d) with $c \neq d$.

* Lesson we learn : "Mirroring" is a good strategy to keep in mind.

Q : What about starting states with > 2 non-empty piles?

E.g. $(1,2,3)$

Observe: It's not in P1's favour to eat any single pile

We're left with:

$(1,2,3)$ → (all N)

N $(1,1,3)$ $\quad$ $(1,2,2)$ N $\quad$ $(1,2,1)$ N

$\Rightarrow$ $(1,2,3)$ is P.

(Note: $(1,1,3)$ is N because there is a move to $(1,1,0)$, which is P)

E.g. $(1,2,4)$

$(1,2,4)$ N

$(1,1,4)$ $\quad$ $(1,2,3)$ P $\quad$ $(1,2,2)$ $\quad$ $(1,2,1)$ $\quad$ ...

$(1,2,4)$ is N because it has a move to $(1,2,3)$, which is P.

Let's work towards a solution

Given a position $(m_1, m_2, \ldots, m_k)$ we're going to compute its "nim-sum": $m_1 \oplus m_2 \oplus \cdots \oplus m_k$.

= decimal corresponding to the XOR of the binary representation of $m_1, m_2, m_3, \ldots m_k$.

The nim-sum is defined in terms of XOR = exclusive or.

Given some m, we can convert it to binary.

E.g. $m = 5 = \underline{1} \cdot 2^2 + \underline{0} \cdot 2^1 + \underline{1} \cdot 2^0 = 2^2 + 1$

$5 = (101)_2$

Expand m as a sum of distinct powers of 2, something like $a_k \cdot 2^k + a_{k-1} \cdot 2^{k-1} + a_{k-2} 2^{k-2} + \cdots + a_0 2^0$

where each a_i is either 1 or 0.

Then $m = (a_k a_{k-1} \cdots a_0)_2$

E.g. $19 = 16 + 3 = 16 + 2 + 1 = 1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2 + 1 \cdot 2^0$

$19 = (10011)_2$

E.g. (nim sum) of $(5, 7, 19)$

$$
\begin{array}{rcr}
5 = & & 101_2 \\
\oplus\ 7 = & & 111_2 \\
\oplus\ 19 = & \oplus & 10011_2 \\
\hline
17 & & 10001_2
\end{array}
$$

(column-wise XOR)

$$
\begin{bmatrix}
1 \oplus 1 = 0 \\
1 \oplus 0 = 1 \\
0 \oplus 1 = 1 \\
0 \oplus 0 = 0
\end{bmatrix}
$$