MATH 2301 17 Oct 2022

* Today : Nim

Game state = some number of piles/heaps each consisting of some number (≥ 0) of berries.
(order is irrelevant, but keep track of multiplicity)

A move consists of eating a non-zero number of berries from exactly one pile.

E.g : (1, 2, 2) ~~≠ (1,2,2,1,2,2)~~ (game states are tuples) ~~up to reordering~~

(0,2,2) (1,1,2) (1,0,2) ...

The game ends when there are no moves.

* Easy cases
- If $\exists$ exactly one pile of a non-zero # of berries then the first player wins.

$\Rightarrow$ Any position of the form (m), for ~~...~~ $m > 0$, is an N-position.

- Positions of the form (a, b) for $a, b > 0$

N (5, 8) or (4, 4) P

(4, 4) → (2, 4) → (2, 2) → ...

* Claim: A position of the form (a, b) where $a, b \geq 0$ and $a \neq b$ is an N-position

A position of the form (a, b) with $a = b$ is a P-position.

Pf (sketch): If (a, b) is such that ~~always~~ $a \neq b$ and $a, b \geq 0$, then, suppose (WLOG) that $a > b$

The winning move is

$(a, b) \longmapsto (b, b)$

Subsequently, no matter what the second player does, the first player can "mirror" the move.

E.g. $(4, 7) \to (4, 4) \to (1, 4) \to (1, 1)$

Similarly, from a position of the form (a, a), the only possible moves go to positions of the form (c, d) with $c \neq d$.

* Lesson we learn: "Mirroring" is a good strategy to keep in mind.

Q: What about starting states with > 2 non-empty piles?

E.g. (1, 2, 3)

Observe: It's not in P1's favour to eat any single pile

We've left with:

(1, 2, 3) → (all N)

N (1, 1, 3) (1, 2, 2) N (1, 2, 1) N

$\Rightarrow$ (1, 2, 3) is P.

(Note: (1, 1, 3) is N because there is a move to (1, 1, 0), which is P)

E.g. (1, 2, 4)

(1, 2, 4) N → …

(1, 1, 4) (1, 2, 3) P (1, 2, 2) (1, 2, 1)

(1, 2, 4) is N because it has a move to (1, 2, 3), which is P.

Let's work towards a solution

Given a position $(m_1, m_2, \ldots, m_k)$ we're going to compute its "nim-sum": $m_1 \oplus m_2 \oplus \cdots \oplus m_k$.

= decimal corresponding to the XOR of the binary representation of $m_1, m_2, m_3, \ldots m_k$.

The nim-sum is defined in terms of XOR
= exclusive or.

Given some m, we can convert it to binary.

E.g. $m = 5 = \underline{1} \cdot 2^2 + \underline{0} \cdot 2^1 + \underline{1} \cdot 2^0 = 2^2 + 1$

$$5 = (101)_2$$

Expand m as a sum of distinct powers of 2,
something like $a_k \cdot 2^k + a_{k-1} \cdot 2^{k-1} + a_{k-2} 2^{k-2} + \cdots + a_0 2^0$

where each a_i is either 1 or 0.

Then $m = (a_k a_{k-1} \cdots a_0)_2$

E.g. $19 = 16 + 3 = 16 + 2 + 1 = 1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2 + 1 \cdot 2^0$

$$19 = (10011)_2$$

E.g. (nim sum) of $(5, 7, 19)$

$$\begin{array}{rcr} 5 & = & 101_2 \\ \oplus\ 7 & = & 111_2 \\ \oplus\ 19 & = \oplus & 10011_2 \\ \hline 17 & & 10001_2 \end{array}$$

(column-wise XOR)

$$\begin{bmatrix} 1 \oplus 1 = 0 \\ 1 \oplus 0 = 1 \\ 0 \oplus 1 = 1 \\ 0 \oplus 0 = 0 \end{bmatrix}$$