MATH 2301 19 Oct 2022

** Last time : Nim

** Today : Winning strategy (?) using nim-sum.

Recall: The nim-sum of $(m_1, \dots, m_k)$ is the (decimal representation of) the column-wise XOR of the binary representations of $m_1, m_2, \dots, m_k$.

Example: $(4, 9, 11)$

$$4 = 2^2 = 1\cdot 2^2 + 0\cdot 2^1 + 0\cdot 2^0 = 100_2$$
$$9 = 8+1 = 2^3+1 = 1\cdot 2^3 + 0\cdot 2^2 + 0\cdot 2^1 + 1\cdot 2^0 = 1001_2$$
$$11 = 8+2+1 = 1\cdot 2^3 + 0\cdot 2^2 + 1\cdot 2^1 + 1\cdot 2^0 = 1011_2$$

$$\begin{array}{rcr} 4 & & 100_2 \\ \oplus\ 9 & = & \oplus\ 1001_2 \\ \oplus\ 11 & & \oplus\ 1011_2 \\ \hline 6 & = & 0110_2 \end{array}$$

** Properties of nim-sum

- Commutative: $x \oplus y = y \oplus x$
- Associative: $x \oplus (y \oplus z) = (x \oplus y) \oplus z$
- $x \oplus x = 0$ for any x.
- In fact, $x \oplus y = 0$ if and only if $x = y$.

$\Rightarrow$ If ⊛ $x \oplus y = z$ then

$y = x \oplus z$ [by adding x to both sides of ⊛]

and $x = y \oplus z$ [by adding y to both sides of ⊛]

- $(x_1 \oplus x_2 \oplus \dots \oplus x_k)$ in binary puts a "1" in each column of the binary XOR that has an odd number of "1"s, and a "0" in each column that has an even number of "1"s.

* Theorem: Suppose that $(m_1, m_2, \dots, m_k)$ is a game state in a game of nim. This state is an "N" state if and only if $(m_1 \oplus m_2 \oplus \dots \oplus m_k) > 0$.
Otherwise, if the nim-sum is zero, it is a P state (and conversely).

* Examples

- Game state (m), the nim sum is m. So it is "N" if and only if $m > 0$.

- Game state (a, b), the nim sum is $(a \oplus b)$. It is "N" if and only if $(a \oplus b) > 0$, which occurs if and only if $a \neq b$.

- State $(1, 2, 3)$ → it is a P state.

$$1 \oplus 2 \oplus 3 = 1_2 \oplus 10_2 \oplus 11_2 = \begin{array}{r} 1_2 \\ 10_2 \\ 11_2 \\ \hline 00_2 \end{array} = 0$$

- State $(1, 2, 4)$ → it is N

$$1 \oplus 2 \oplus 4 = \begin{array}{r} 1_2 \\ \oplus\ 10_2 \\ \oplus\ 100_2 \\ \hline 111_2 \end{array} = 7 > 0.$$

The theorem implies the following statements:

① If $m_1 \oplus \cdots \oplus m_k = 0$, then any move results in a positive nim-sum. [by property of P-states]

② If $m_1 \oplus \cdots \oplus m_k > 0$, then there is at least one move that results in a zero nim-sum.
[by property/def. of N-states]

Let us check these.

① Suppose $(m_1, \ldots, m_k)$ is a state such that [E.g. (1,2,3)]

⊛ $\boxed{m_1 \oplus \cdots \oplus m_k = 0}$.

Suppose we eat some berries out of the first pile. The new state will be
~~[illegible]~~ $(m_1', m_2, \ldots, m_k)$

The new nim-sum is
$m_1' \oplus \underline{m_2 \oplus \cdots \oplus m_k}$

<u>Note</u>: $(m_2 \oplus \cdots \oplus m_k) = m_1$ [by adding m_1 to both sides of ⊛]

$\Rightarrow$ new nim-sum is
$m_1' \oplus m_1$. Since $m_1' \neq m_1$, we see that

$m_1' \oplus m_1$ = new nim sum > 0

The same argument works for making moves in any other pile $\Rightarrow$ any move sends you to a state with a positive nim-sum.

② Let's check the other statement.
Suppose $(m_1, \ldots, m_k)$ is a state such that
$m_1 \oplus \cdots \oplus m_k > 0$.
We need to find at least one move that makes the nim-sum zero.

Let $\boxed{S = m_1 \oplus \cdots \oplus m_k}$

Suppose we make a move in the i^{th} pile and suppose also that the new nim-sum is zero.
$m_i \mapsto m_i'$

$$\boxed{0 = m_1 \oplus m_2 \oplus \cdots \oplus m_{i-1} \oplus m_i' \oplus m_{i+1} \oplus \cdots \oplus m_k}$$

Adding the two equations, we see:
$S = (\cancel{m_1} \oplus \cdots \oplus \cancel{m_k}) \oplus (\cancel{m_1} \oplus \cdots \oplus m_i' \oplus \cdots \oplus \cancel{m_k})$

$\boxed{S = m_i \oplus m_i'}$

$\Rightarrow \boxed{m_i' = m_i \oplus S}$

We just have to make sure that there is an i such that $(m_i \oplus S) < m_i$

[to be continued...]