

* Last time:

Theorem: Let (m_1, \dots, m_k) be a nim position.

Consider $s = m_1 \oplus \dots \oplus m_k$.

Then this is an N position iff $s > 0$

and a P-position iff $s = 0$.

Theorem implies:

- ① If $s = 0$, then any move makes the new s positive. [All moves out of a P-state go to N-states]
- ② If $s > 0$, then there is a move that makes the new s zero. [Each N-state has at least one move to a P state.]

Last time: We checked ① and part of ②

* Today: We finish ②

Consider (m_1, \dots, m_k) a game state with $s = m_1 \oplus \dots \oplus m_k$, and $s > 0$.

Suppose we make a move $m_i \mapsto m_i'$.

Suppose also that

$$0 = m_1 \oplus m_2 \oplus \dots \oplus m_{i-1} \oplus m_i' \oplus m_{i+1} \oplus \dots \oplus m_k$$

Add the two equations:

$$s \oplus 0 = m_i \oplus m_i' = s \Rightarrow m_i' = s \oplus m_i$$

This tells us what our ~~new~~ move has to be, provided that $m_i' = (s \oplus m_i) < m_i$.

To check what we need, we need to ensure that $(s \oplus m_i) < m_i$ for at least one of the ~~old~~ piles.

Example. $(4, 5, 7)$

$$\begin{array}{r} 4 \oplus 5 \oplus 7 = 100_2 \oplus 101_2 \oplus 111_2 = 110_2 = 6 \\ m_1 \quad m_2 \quad m_3 \quad s \end{array}$$

$$\text{In this case, } 100_2 \oplus 110_2 = m_1 \oplus s = 10_2 = 2 < 4$$

$$101_2 \oplus 110_2 = m_2 \oplus s = 11_2 = 3 < 5$$

$$111_2 \oplus 110_2 = m_3 \oplus s = 1_2 = 1 < 7$$

$\Rightarrow \exists$ Three possible moves that send you to a P-position.

Example: ~~(4, 5, 7)~~ $(10, 13, 12, 8)$

$$\begin{array}{r} 10 = 1010_2 \leftarrow m_1 \\ 13 = 1101_2 \leftarrow m_2 \\ 12 = 1100_2 \leftarrow m_3 \\ \oplus 8 = 1000_2 \leftarrow m_4 \\ \hline 3 = 0011_2 \leftarrow s \end{array}$$

Claim: \exists one winning move, namely $10 \mapsto (10 \oplus 3)$

This worked because the leftmost binary digit of $s = 3$ cancelled with a corresponding "1" in the binary representation of $10 = 1010_2$.
 $\Rightarrow (10 \oplus 3) < 10$, because the binary rep of $10 \oplus 3$ equals the binary rep of 10 in the columns before the "1" that cancelled, and moreover a "1" in the next column got cancelled.

⇒ To be able to successfully make a ^(winning) move, we can do so in any m_i ~~we wish~~ which has a "1" in the column corresponding to the leftmost column of s (ignoring leading "0"s), in the binary representation.

Observe: The leftmost column of s (ignoring leading "0"s) is precisely the first column from the left that has an odd number of "1"s.

Such a column exists because $s \neq 0$

⇒ There is at least one m_i that has a "1" in that column, so you can make at least one winning move!

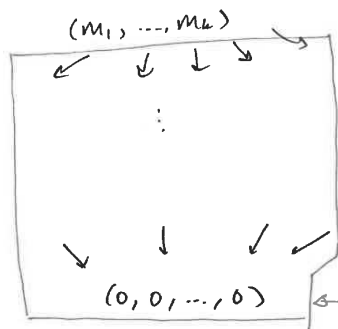
Upshot: If $m_1 \oplus \dots \oplus m_k > 0$, then we know how to win!

If $m_1 \oplus \dots \oplus m_k = 0$, then "

The arguments we just gave to check ① & ② can be turned into a proof of the theorem.

Sketch: Look at a position (m_1, \dots, m_k) & its game graph.

Idea: induct bottom-to-top on game graph.



← assume this portion satisfies the theorem, and conclude for (m_1, \dots, m_k) .

← P has nim-sum 0

* Recall: N/P labelling of a game graph [sometimes called "strategic labelling"]

Next week: Upgrade N/P labelling to the "Grundy labelling".

