MATH 2301 20 Oct 2022

★ Last time:

Theorem: Let $(m_1, \dots, m_k)$ be a nim position. Consider $s = m_1 \oplus \dots \oplus m_k$.
Then this is an N position iff $s > 0$ and a P-position iff $s = 0$.

Theorem implies:

1. If $s = 0$, then any move makes the new s positive. [All moves out of a P-state go to N-states]
2. If $s > 0$, then there is a move that makes the new s zero. [Each N-state has at least one move to a P state.]

Last time: we checked 1 and part of 2

* Today: We finish 2

Consider $(m_1, \dots, m_k)$ a game state with $\boxed{s = m_1 \oplus \dots \oplus m_k}$, and $s > 0$.

Suppose we make a move $m_i \mapsto m_i'$.
Suppose also that

$$\boxed{0 = m_1 \oplus m_2 \oplus \dots \oplus m_{i-1} \oplus m_i' \oplus m_{i+1} \oplus \dots \oplus m_k}$$

Add the two equations:

$s \oplus 0 = \boxed{m_i \oplus m_i' = s.} \Rightarrow \boxed{m_i' = s \oplus m_i}$

This tells us what our move has to be, provided that $m_i' = (s \oplus m_i) < m_i$.

To check what we need, we need to ensure that

$(s \oplus m_i) < m_i$ for at least one of the piles.

Example. $(4, 5, 7)$

$$\underset{m_1}{4} \oplus \underset{m_2}{5} \oplus \underset{m_3}{7} = 100_2 \oplus 101_2 \oplus 111_2 = 110_2 = \underset{s}{6}.$$

In this case,

$$100_2 \oplus 110_2 = m_1 \oplus s = 10_2 = 2 < 4$$
$$101_2 \oplus 110_2 = m_2 \oplus s = 11_2 = 3 < 5$$
$$111_2 \oplus 110_2 = m_3 \oplus s = 1_2 = 1 < 7$$

$\Rightarrow \exists$ three possible moves that send you to a P-position.

Example: $(10, 13, 12, 8)$

$$\begin{array}{rrcl}
 & 10 & = & 1\,0\,\boxed{1}\,0_2 \leftarrow m_1 \\
 & 13 & = & 1\,1\,0\,1_2 \leftarrow m_2 \\
 & 12 & = & 1\,1\,0\,0_2 \leftarrow m_3 \\
\oplus & 8 & = & 1\,0\,0\,0_2 \leftarrow m_4 \\
\hline
 & 3 & = & 0\,0\,1\,1_2 \leftarrow s
\end{array}$$

Claim: $\exists$ one winning move, namely $10 \mapsto \underset{9}{\underset{\shortparallel}{(10 \oplus 3)}}$

This worked because the leftmost binary digit of $s = 3$ cancelled with a corresponding "1" in the binary representation of $10 = 1010_2$

$\Rightarrow (10 \oplus 3) < 10$, because the binary rep of $10 \oplus 3$ equals the binary rep of 10 in the columns before the "1" that cancelled, and moreover a "1" in the next column got cancelled.

$\Rightarrow$ To be able to successfully make a (winning) move, we can do so in any m_i ~~one which~~ ~~the~~ which has a "1" in the column corresponding to the leftmost column of s (ignoring leading "0"s), in the binary representation.

Observe: The leftmost column of s (ignoring leading "0"s) is precisely the first column from the left that has an odd number of "1"s.

Such a column exists because $s \neq 0$

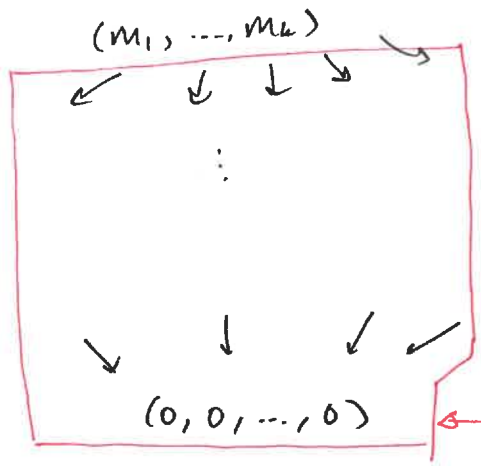
$\Rightarrow$ There is at least one m_i that has a "1" in that column, so you can make at least one winning move!

Upshot: If $m_1 \oplus \cdots \oplus m_k > 0$, then we know how to win!

If $m_1 \oplus \cdots \oplus m_k = 0$, then ☹

The arguments we just gave to check 1 & 2 can be ~~x~~ turned into a proof of the theorem.

Sketch: Look at a position $(m_1, \ldots, m_k)$ & its game graph.



Idea: induct bottom-to top on game graph.

assume this portion satisfies the theorem, and conclude for $(m_1, \ldots, m_k)$.

P has nim-sum 0

* Recall: N/P labelling of a game graph
[sometimes called "strategic labelling".]

Next week: Upgrade N/P labelling to the "Grundy labelling".