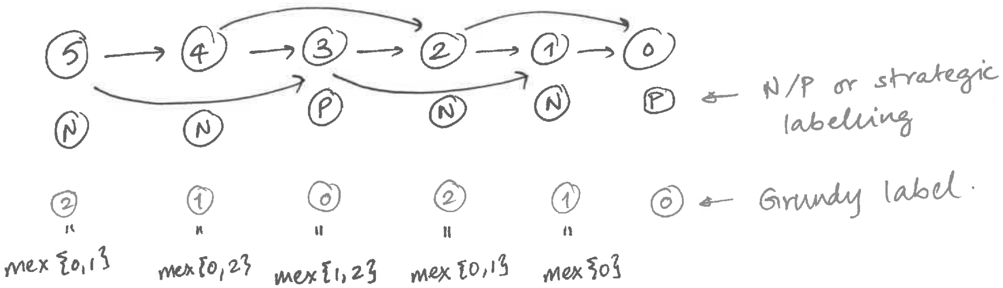
MATH 2301 24 Oct 2022

* Final exam info is on Wattle.

* Today: Grundy labelling

Example: ① subtraction game

$n = 5$, $S = \{1, 2\}$



Steps:

① Label terminal positions by 0

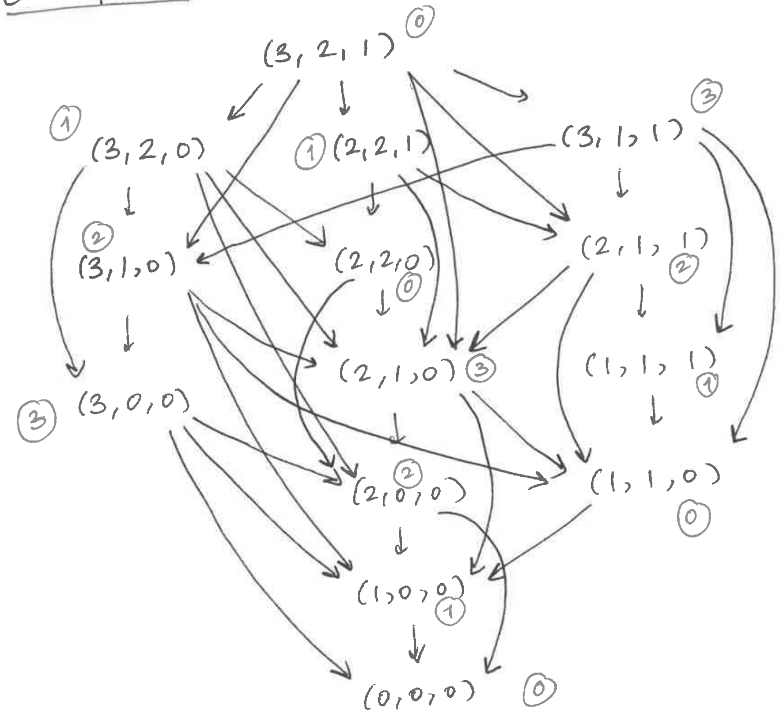
② Next, label any state such that ~~any~~ every place reachable from it already has a label.

③ Suppose that the state you're trying to label has outgoing arrows to states labelled $\{n_1, n_2, \dots, n_k\}$. This state then gets labelled by

$$\text{mex}\{n_1, \dots, n_k\}$$

↑ "minimum excluded" → see examples (starting from 0)

Example 2 (Nim)



Theorem: If $(m_1, \dots, m_k)$ is a nim game position, then its Grundy label is precisely its nim-sum $m_1 \oplus \dots \oplus m_k$.

Pf sketch: Let $(m_1, \dots, m_k)$ be a nim position. We'll show that $s = (m_1 \oplus \dots \oplus m_k)$ is precisely the mex of all Grundy labels of states that are reachable in one step from $(m_1, \dots, m_k)$

[By induction on the graph, assume that for any state that $(m_1, \dots, m_k)$ points to, the nim-sum equals the Grundy label.]

We need to show:

① No move takes us to a state with nim-sum $= s$.

② For any $s' < s$, there is a move that takes us to a state with nim-sum s'.

Let's look at ①

We're at $(m_1, \dots, m_k)$. If we make any move $m_i \mapsto m_i'$, then the new nim-sum is

$m_1 \oplus \dots \oplus m_{i-1} \oplus m_i' \oplus \dots \oplus m_k = s'$, whereas

$m_1 \oplus \dots \oplus m_{i-1} \oplus m_i \oplus \dots \oplus m_k = s$

Get $\boxed{m_i' \oplus m_i = s' \oplus s}$.

Note: $m_i' \neq m_i \Rightarrow s' \neq s$

Let's look at ②

Let $s' < s$, we want to find a move $m_i \mapsto m_i'$ such that the new nim-sum is s'.

As before, $m_i' \oplus m_i = s' \oplus s$

$\Rightarrow \boxed{m_i' = s' \oplus s \oplus m_i}$

Need some i such that $m_i' < m_i$

$$
\begin{array}{rcl}
s_2 & = & 1 * \dots \boxed{1} * * \dots * \\
\oplus \quad s_2' & = & 1 * \dots \boxed{0} * \dots * \\
\hline
(s' \oplus s)_2 & = & 0\,0\,0 \dots 1 * * *
\end{array}
$$

the first column from the left where s & s' differ; a "1" in s becomes a "0" in s'.

$m_i = (s' \oplus s) \oplus m_i'$

$\exists$ at least one i that has a "1" in the same column as the initial column of $(s' \oplus s)$

By the same argument as last week, there is some i for which we can make a move.

$\square$.