MATH 2301 26/10/2022

* Last time: Grundy label & in nim, the Grundy label of a position $(m_1, \dots, m_k)$ is simply $m_1 \oplus \cdots \oplus m_k$.

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* A few words about Grundy labelling.

Theorem: Let G be any game position (in any impartial combinatorial game) with Grundy label g. Then, G is an N-state if $g > 0$, and a P-state if $g = 0$.

Pf sketch: We need to check:

1. If $g > 0$, then there exists a move to a state with Grundy label 0.
2. If $g = 0$, then every move from G goes to a state with Grundy label positive.

These are true by the mex definition of Grundy labels.

[Exercise!]

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* Main focus of today: Sums of games

Let G and H be game states of two (not necessarily the same) impartial combinatorial games. [E.g. nim / chomp / hackenbush, etc.]

* Def: Under the hypotheses above, we say that $(G+H)$ defines a game state for a new game, with the following rules:

- a move consists of making a single move either in G or in H, but not both; ie:
  $(G, H) \mapsto (G', H)$ or $(G, H) \mapsto (G, H')$
- the player who can't make a move in either game loses.

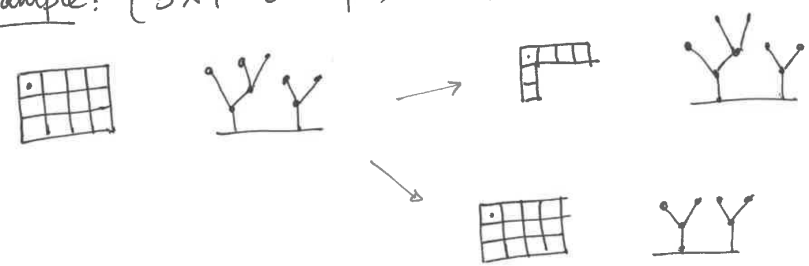
Example: Nim, again!

(Notation: The nim game state with a single pile of n berries is denoted $(*n)$.)

Then the nim game position $(m_1, m_2, \dots, m_k)$ is just $\underline{(*m_1) + (*m_2) + (*m_3) + \cdots + (*m_k)}$

↑ Addition as described above, for game states.

Example: (3×4 chomp) + (hackenbush)



[Two possible moves]

Q: What happens to $G+H$ if you know the N/P label of G & H individually?

Some cases:

1. Both G & H are P states.

<u>Claim</u>: $G+H$ is P as well.

[If P1 makes a move in G, then P2 can counter it with another move in G that sends it back to a P state. Similarly for H.] ⇒ ~~not~~ no way to force a win for the 1st player.]

2. Say G is N state and H is a P state.

<u>Claim</u>: $G+H$ is an N state.

[P1 can make a move in G to send & it to a P state, and use previous case to see that (P, P) is losing for P2.]

3. Both G & H are N positions

<u>Claim</u>: The N/P status of $G+H$ is inconclusive

<u>E.g. 1</u>: $G = (*7)$, $H = (*7)$ ⇒ $G+H = (*7)+(*7)$ is a P state.
N N

<u>E.g. 2</u>: $G = (*7)$, $H = (*8)$ ⇒ $G+H = (*7)+(*8)$ is an N-state.

~~Theorem~~

<u>Rmk</u>: Not always possible to deduce the N/P label of $G+H$ from the N/P labels of G & H.

<u>Theorem</u>: It is possible to deduce the Grundy label of $G+H$, given the Grundy labels of G & H.

Suppose g & h are Grundy labels of G & H. Then, the Grundy label of $G+H$ is simply $(g \oplus h)$.

<u>Pf idea (vague)</u>: Same idea that proves that the Grundy label of a nim position $(m_1, \dots, m_k)$ is $(m_1 \oplus \dots \oplus m_k)$.

[(Hard) exercise, we'll skip the details.]

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* <u>First look at equivalence of games</u>

<u>Def</u>: Let G and G' be two game states (of possibly different games). We say that $G \sim G'$ if: for any ~~other~~ game state, H, either:

$(G+H)$ and $(G'+H)$ are both N-states, or

$(G+H)$ and $(G'+H)$ are both P-states.

[<u>Exercise</u>: Check that this is an equivalence relation!]