MATH 2301 26/10/2022

* Last time: Grundy label of m nim, the Grundy label of a position $(m_1, \dots, m_k)$ is simply $m_1 \oplus \dots \oplus m_k$.

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* A few words about Grundy labelling.

Theorem: Let G be any game position (in any impartial combinatorial game) with Grundy label g. Then, G is an N-state if $g > 0$, and a P-state if $g = 0$.

Pf sketch: We need to check:

(1) If $g > 0$, then there exists a move to a state with Grundy label 0.

(2) If $g = 0$, then every move from G goes to a state with Grundy label positive.

These are true by the mex definition of Grundy labels.

[Exercise!]

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* Main focus of today: Sums of games

Let G and H be game states of two (not necessarily the same) impartial combinatorial games. [E.g. nim / chomp / hackenbush, etc.]

* Def: Under the hypotheses above, we say that $(G+H)$ defines a game state for a new game, with the following rules:

- a move consists of making a single move either in G or in H, but not both; ie:

  $(G,H) \mapsto (G',H)$ or $(G,H) \mapsto (G,H')$

- the player who can't make a move in either game loses.

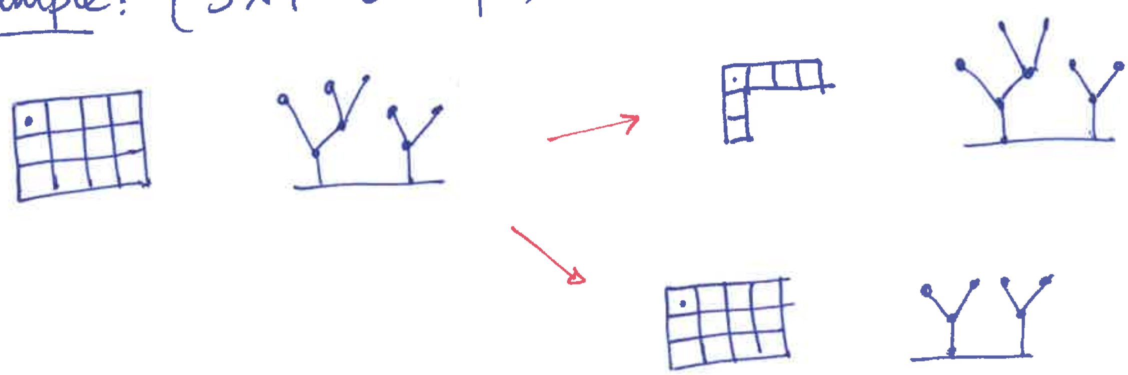
Example: Nim, again!

(Notation: The nim game state with a single pile of n berries is denoted $(*n)$.)

Then the nim game position $(m_1, m_2, \dots, m_k)$ is just $(*m_1) + (*m_2) + (*m_3) + \dots + (*m_k)$

Addition as described above, for game states.

Example: (3×4 chomp) + (hackenbush)



[Two possible moves]

Q: What happens to G+H if you know the N/P label of G & H individually?

Some cases:

1. Both G & H are P states.
Claim: G + H is P as well.
[If P1 makes a move in G, then P2 can counter it with another move in G that sends it back to a P state. Similarly for H. } ⇒ no way to force a win for the 1st player.]

2. Say G is N state and H is a P state.
Claim: G+H is an N state.
[P1 can make a move in G to send it to a P state, and use previous case to see that (P, P) is losing for P2.]

3. Both G & H are N positions
Claim: The N/P status of G+ H is inconclusive
E.g. 1: $G = (*7)$, $H = (*7)$ ⇒ $G+H = (*7)+(*7)$ is a P state.
N N

E.g. 2: $G = (*7)$, $H = (*8)$ ⇒ $G+H = (*7)+(*8)$ is an N-state.

~~Theorem~~

Rmk : Not always possible to deduce the N/P label of G+H from the N/P labels of G. & H.

Theorem: It is possible to deduce the Grundy label of G+H, given the Grundy labels of G & H. Suppose g & h are Grundy labels of G & H. Then, the Grundy label of G+H is simply

$(g \oplus h)$.

Pf idea (vague): Same idea that proves that the Grundy label of a nim position $(m_1, \ldots, m_k)$ is $(m_1 \oplus \cdots \oplus m_k)$.
[(Hard) exercise, we'll skip the details.]

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* First look at equivalence of games

Def: Let G and G' be two game states (of possibly different games). We say that $G \sim G'$ if: for any ~~other~~ game state, H, either:
$(G+H)$ and $(G'+H)$ are both N-states, or
$(G+H)$ and $(G'+H)$ are both P-states.

[Exercise: check that this is an equivalence relation!]