MATH 2301 27/10/2022

* Recap: Let G & G' be two game positions (possibly different games)

We say $G \sim G'$ if for any game position H, the positions $(G+H)$ and $(G'+H)$ have the same outcome [either both N, or both P].

[Exercise: Check that this is an equivalence relation.]

Prop: If $G \sim G'$, then G & G' have the same outcome.

Pf: Let H be an "empty game", ie, a terminal position of your favourite game.

Then $G+H$ and $G'+H$ have the same outcome

But no moves are possible in H, so G and G' have the same outcome.

Warning: If G and G' have the same outcome, Even they may not be equivalent.

E.g. Let G be N and G' also an N position.

$G = (*10)$, $G' = (*22)$

Let $H = *10$

$(G+H) = (*10) + (*10) \to$ P position

$(G'+H) = (*22) + (*10) \to$ N position !

* Prop: Let G be any position. Let L be a P-position.

Then, $G \sim (G+L)$.

Pf (sketch): Let H be any other position. Need to show that $G+H$ and $(G+L)+H$ have the same outcome.

Suppose that $G+H$ is an N position. We need to find a winning strategy for $(G+L+H)$ (or otherwise show that $G+L+H$ is an N-position.)

Consider $G+L+H$ as $(G+H)+L$

As yesterday, you can make a move in $G+H$ that sends $G+H$ to a P-state, and then the second player gets a sum of two P-games.

Alternatively, look at Grundy labels:

If g, h, ℓ are the Grundy labels of G, H, L,

then: ① $g \oplus h > 0$

② $\ell = 0$

$\Rightarrow g \oplus h \oplus \ell = g \oplus h > 0$.

Corollary: Let L and L' be two P-positions. Then $L \sim L'$.

Pf: Let H be any game position with Grundy label h. The Grundy values of L and L' are both zero. So, $L+H$ and $L'+H$ both have Grundy labels $(0 \oplus h) = h$.

[All P-positions are equivalent!]

Theorem: Let G and G' be two game positions. Then $G \sim G'$ if and only if $G+G'$ is a P-position.

Pf: ($\Rightarrow$) Let $G \sim G'$. Then we'll show that $G+G'$ is a P-position.

Take $H = G$. Then $G+G$ and $G'+G$ have the same outcome.

Note: $G+G$ is a P-position (see this, e.g. using a mirroring strategy, or Grundy values.)

So $G+G'$ is also a P-position!

($\Leftarrow$) Now suppose $G+G'$ is a P-position. Let us try to show that $G \sim G'$.

[Strategy 1: Let H be any game. Derive a contradiction if we assume that $G+H$ and $G'+H$ have different outcomes.]

If $(G+H)$ is N and $(G'+H)$ is P, then $(G+H)+(G'+H)$ is N (from yesterday)

But $(G+H)+(G'+H) = (G+G') + \underbrace{(H+H)}_{\text{P-position}}$

$\Rightarrow \underbrace{(G+G')+(H+H)}_{N!} \sim \underbrace{(G+G')}_{P!}$ (using previous proposition)

This is a contradiction!

Strategy #2: $(G+G')$ is P

By previous prop, $G \sim G + (G+G')$

$G+(G+G') = \underbrace{(G+G)}_{P} + G' \Rightarrow$

$G+(G+G') \sim G'$

$\Rightarrow G \sim G'$

Theorem [Sprague-Grundy]: Every game position G in any impartial combinatorial game is equivalent to a single-pile nim position.

In fact if $g =$ Grundy value of G, then $G \sim (*g)$.

Pf: $G+(*g)$ has a Grundy value of $g \oplus g = 0 \Rightarrow G+(*g)$ is a P-state.

Now use the previous theorem!