MATH 2301 — 31/07/2023

* Admin :
  - Reflective Check-in 1 → due Tuesday this week, Sunday all other weeks. } Wattle
  - Gradescope – still a work in progress, sorry!
  - HW 1 due Friday → up on Wattle → Assignments
  - Workshops!

* Last time : Equivalence relations
  - reflexivity, symmetry, transitivity.

Today : Equivalence classes & more.

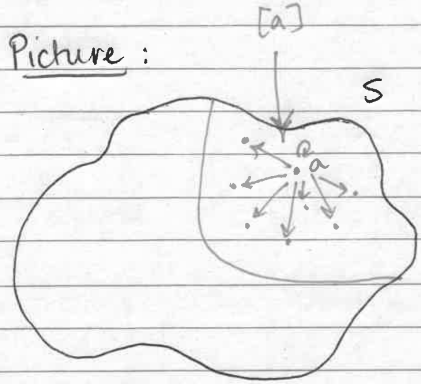
Let S be a set and R is an equivalence reln on S.

Defn : If $a \in S$, then the equivalence class of a (with respect to the equiv. reln. R) is the set
$$\{ b \in S \mid (a,b) \in R \}, \text{ denoted by } [a]_R$$
(or just $[a]$ if there is no confusion).

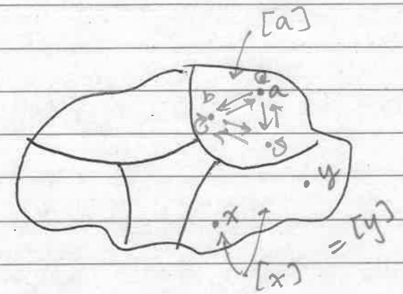
Note : $[a]$ is a subset of S.
also, $[a] \neq \emptyset$ and $a \in [a]$.

Aside!
Q: What does it mean to have an equivalence relation/classes on the empty set?

Picture :



← by def, collect all b such that $(a,b) \in R$.



Actually, the properties of R as an equiv. reln guarantee many other connections

* Proposition

1) If $b, c \in [a]$ then $(b,c) \in R$
2) If $y \in [x]$ then $x \in [y]$ and $[x] = [y]$
3) If $(y,z) \in R$ and $z \in [x]$ then $y \in [x]$
4) If E_1 and E_2 are two equiv. classes, then either $E_1 = E_2$, or $E_1 \cap E_2 = \emptyset$.

(Proof → pdf notes)

Summary: R cuts up S into pieces (subsets) which don't overlap, and within each subset, any two elements are connected.

and there are no connections across subsets.

Example : $S = \mathbb{Z}$ = set of integers

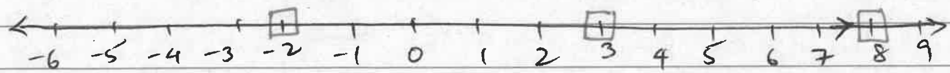
1) $R_2 = \{(x,y) \in \mathbb{Z}\times\mathbb{Z} \mid x-y \text{ is even}\}$
→ an equiv relation.

odds $= [3] = \{\ldots, -3, -1, 1, 3, \ldots\} = [101] = [-57]$

evens $= [2] = \{\ldots, -2, 0, 2, \ldots\} = [6] = [-100]$

Rmk : This relation R_2 has two distinct equiv. classes.

2) $R_5 = \{(x,y) \in \mathbb{Z}\times\mathbb{Z} \mid x-y \text{ is divisible by } 5\}$



$[-2] = \{\ldots, -12, -7, -2, 3, 8, 13, \ldots\}$

Rmk : There are 5 different equiv classes, namely classified by the remainder when you divide by 5.

Let d be any positive integer.

$R_d = \{(x,y) \in \mathbb{Z}\times\mathbb{Z} \mid (x-y) \text{ is divisible by } d\}$

ie, x and y have the same remainder when you divide them by d.

Possibilities for remainder are $0, 1, 2, \ldots, (d-1)$.

⇒ there are exactly d different equiv. classes.

Notation : If R is an equiv. relation and $(x,y) \in R$, we say $x \sim_R y$.
read as x is equivalent to y w.r.t R.

(or we'll just write $x \sim y$ if R is clear)

Back to R_d : If $x \sim_{R_d} y$, write more concisely as $x \sim_d y$.

i.e. $x \sim_{17} y$ means $(x-y)$ is divisible by 17.

Standard labelling of the d possible equiv. classes is:

$[0]_d, [1]_d, \ldots, [d-1]_d$

all integers b such that $(b-1)$ is divisible by d

→ Next time : Modular addition
[an addition-like operation on equivalence classes of R_d]