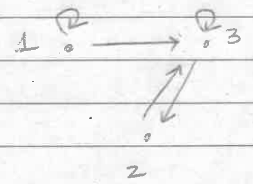
MATH 2301 07/08/2023

* Last time: Matrix operations (esp. products), adjacency matrices of graphs

* Today: Paths/walks in graphs & relationship to adjacency matrix. (for me paths = walk, and either can repeat vertices in the graph)

Example:



$$A = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$(i,j)^{th}$ entry of A is the number of edges from i to j

Consider $A^2 = A \cdot A$

matrix product

$$A^2 = \begin{bmatrix} 1 & 0 & 1 \\ \boxed{0 \quad 0 \quad 1} \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & \boxed{1} \\ 0 & 1 & 2 \end{bmatrix} \end{array}$$

$(i,j)^{th}$ entry of A^2 is $R_i \cdot C_j$

i^{th} row dot j^{th} column

[R_i encodes outgoing edges from i, C_j encodes incoming edges to j]

Proposition: The $(i,j)^{th}$ entry of A^2 is the number of paths of length $= 2$ from i to j in the graph.

Explanation:

$(i,j)^{th}$ entry of A^2

$$= R_i \cdot C_j$$

$$= \cancel{R_{i1} C_{1j} + R_{i2}}$$

$$= A_{i1} A_{1j} + A_{i2} \cdot A_{2j} + \cdots + \underbrace{A_{ik} A_{kj}}_{\text{equals 1 if and only if such edges exist}} + \cdots A_{in} A_{nj}$$

$i \to k \to j$

Theorem: Let ℓ be any ~~positive~~ non-negative integer. The ℓ^{th} power of A, namely A^ℓ has the following property: its $(i,j)^{th}$ entry $A^\ell_{(i,j)}$ is exactly the number of paths from i to j ~~#~~ of length ℓ.

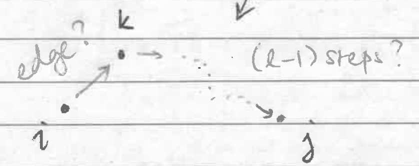
(i.e. # ways to go from i to j in ℓ steps)

Explanation: If ~~l>0~~ $l > 1$, then

$$A^{l} = A \cdot A^{(l-1)}$$

By induction, $A^{(l-1)}$ encodes # paths between vertices of length $(l-1)$.

$$A^{l}_{(i,j)} = A_{i1} A^{(l-1)}_{1j} + A_{i2} \cdot A^{(l-1)}_{2j} \cdots + A_{ik} A^{(l-1)}_{kj} + \cdots + A_{in} A^{(l-1)}_{nj}$$



Range over k, ask: ① is there an edge $i \to k$?
② how many length $(l-1)$ paths dowe have from k to j?
③ take product
$\Downarrow$
$A_{ik} \cdot A^{(l-1)}_{kj}$

Adding over all k takes care of all possibilities.

What about $l = 0$?
By convention, $A^{0} = I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & 1 & & \vdots \\ & & & \ddots & 0 \\ 0 & \cdots & & 0 & 1 \end{pmatrix}$

(i,j) th entry: can you get from i to j without moving?

Q: Let $G = (V, E)$ be a graph and $i, j \in V$.

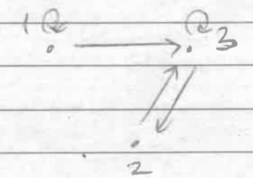
Is there a path from i to j?

Idea: Start adding powers of A } when do we stop??

$$A + A^{2} + A^{3} + \cdots$$

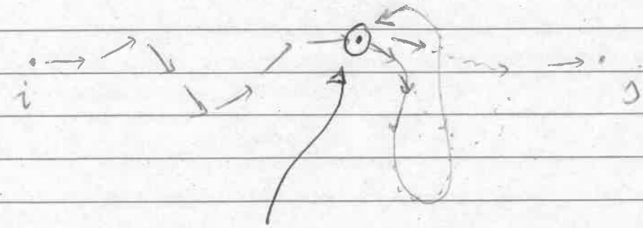
paths of length 1, 2, 3, ...
(we only care about the existence, not how many)

E.g.



Observation: Any long enough path will repeat a vertex.
A path longer than the # vertices is guaranteed to repeat a vertex.

$\Rightarrow$ any path longer than ~~this~~ # vertices has a loop.



at least one repeated vertex in path.

$\Rightarrow$ there is a shorter path from i to j!
[Delete the loop!]