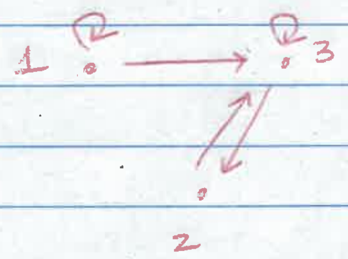
MATH 2301 07/08/2023

* Last time : Matrix operations (esp. products), adjacency matrices of graphs

* Today : Paths/walks in graphs & relationship to adjacency matrix. (for me paths = walk, and either can repeat vertices in the graph)

Example :



$$A = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$(i,j)^{th}$ entry of $A$ is the number of edges from $i$ to $j$

Consider $A^2 = A \cdot A$ ← matrix product

$$A^2 = \begin{bmatrix} 1 & 0 & 1 \\ \boxed{0 \quad 0 \quad 1} \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & \boxed{1} \\ 0 & 1 & 2 \end{bmatrix} \end{array}$$

$(i,j)^{th}$ entry of $A^2$ is $R_i \cdot C_j$

$i^{th}$ row dot $j^{th}$ column.

[$R_i$ encodes outgoing edges from $i$, $C_j$ encodes incoming edges to $j$]

<u>Proposition</u>: The $(i,j)^{th}$ entry of $A^2$ is the number of paths of length $= 2$ from $i$ to $j$ in the graph.

<u>Explanation</u>:

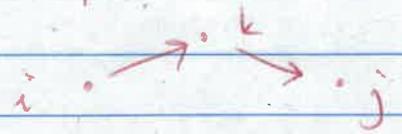
$(i,j)^{th}$ entry of $A^2$

$$= R_i \cdot C_j$$

$$= \cancel{R_{i1} C_{1j} + R_{i2}}$$

$$= A_{i1} A_{1j} + A_{i2} \cdot A_{2j} + \cdots + \underbrace{A_{ik} A_{kj}} + \cdots A_{in} A_{nj}$$

equals 1 if and only if




such edges exist

<u>Theorem</u>: Let $\ell$ be any ~~positive~~ non-negative integer. The $\ell^{th}$ power of $A$, namely $A^{\ell}$ has the following property: its $(i,j)^{th}$ entry $A^{\ell}_{(i,j)}$ is exactly the number of paths from $i$ to $j$ of length $\ell$.

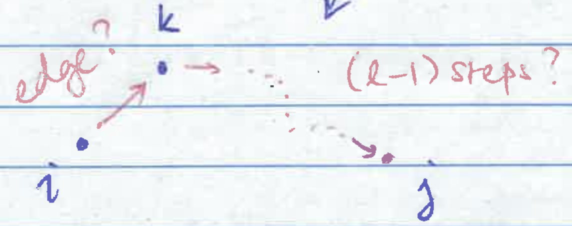
(i.e. # ways to go from $i$ to $j$ in $\ell$ steps)

Explanation : If ~~$\ell \geq 0$ if~~ $\ell > 1$, then

$$A^{\ell} = A \cdot A^{(\ell-1)}$$

By induction, $A^{(\ell-1)}$ encodes # paths between vertices of length $(\ell-1)$.

$$A^{\ell}_{(i,j)} = A_{i1} A^{(\ell-1)}_{1j} + A_{i2} \cdot A^{(\ell-1)}_{2j} \cdots + \underline{A_{ik} A^{(\ell-1)}_{kj}} + \cdots + A_{in} A^{(\ell-1)}_{nj}$$




Range over $k$, ask : 1. is there an edge $i \to k$?
2. how many length $(\ell-1)$ paths dowe have from $k$ to $j$?
3. take product

$$\Downarrow$$

$$A_{ik} \cdot A^{(\ell-1)}_{kj}$$

Adding over all $k$ takes care of all possibilities.

What about $\ell = 0$?

By convention, $A^0 = I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & 1 & & \vdots \\ & & & \ddots & 0 \\ 0 & \cdots & & 0 & 1 \end{pmatrix}$

$(i,j)$ th entry : can you get from $i$ to $j$ without moving?

Q: Let $G = (V, E)$ be a graph and $i, j \in V$.

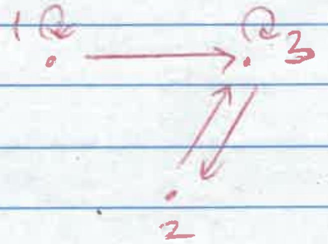
Is there a path from $i$ to $j$?

Idea: Start adding powers of $A$

$A + A^2 + A^3 + \cdots$ } when do we stop??

paths of length 1, 2, 3, ...
(we only care about the existence, not how many)

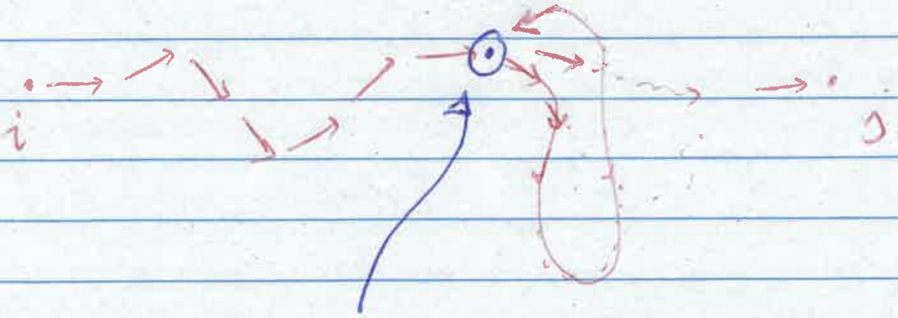
E.g.




Observation: Any long enough path will repeat a vertex.

A path longer than the # vertices is guaranteed to repeat a vertex.

$\Rightarrow$ any path longer than ~~this~~ # vertices has a loop.




at least one repeated vertex in path.

$\Rightarrow$ there is a shorter path from $i$ to $j$!
[Delete the loop!]