MATH 2301 09/08/2023

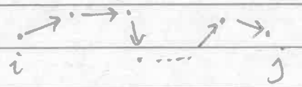
* Directed graphs + adjacency matrices

Paths from i to j? [A = adjacency matrix]

Recall: ~~$A^l_{(i,j)}$~~ $A^{l}_{(i,j)}$ = number of paths of length l from i to j

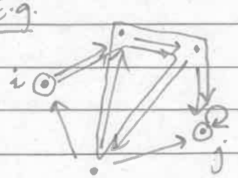
* Question: Is there a path of nonzero length from i to j in a given graph?

* $\boxed{\text{Theorem}}$: Let $G=(V,E)$ be a directed graph with n vertices. Suppose $i, j \in V$ such that there is at least one (non-zero) path from i to j. Then, the shortest path from i to j has ~~at~~ length $\leq n$.

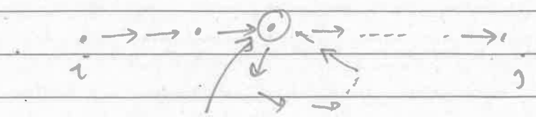


$\boxed{\text{Explanation}}$

E.g.



Take a path from i to j of length $> n$. Since there are n vertices, this path goes through some vertex twice.



→ Skip the loop at the repeated vertex to get a shorter path!

Consequence: Let $G=(V,E)$ have n vertices. Let A be an adjacency matrix for G. Then, to find if there is a path of non-zero length from i to j, we need to compute:

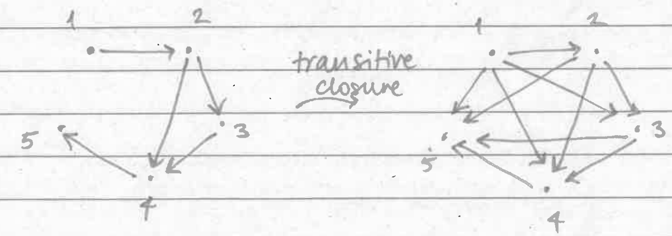
$$(A + A^2 + A^3 + \cdots + A^n)_{(i,j)}$$

→ if this is 0, then there is no path from i to j

→ if this is non-zero, then there is a path from i to j

Rmk: $(A + \cdots + A^m)_{(i,j)}$ is the number of paths from i to j of length between 1 and m, inclusive.

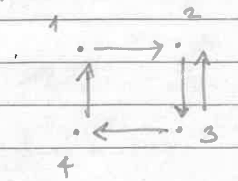
** Transitive closure

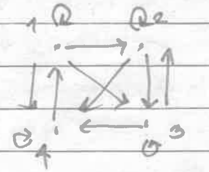


Def: The transitive closure of $R \subseteq S \times S$ is the smallest relation R' such that $R \subseteq R'$ and R' is transitive.

In terms of graphs, to take transitive closure:
- look at any pair (i,j) of vertices such that there is a non-zero path from i to j
- If there is no edge $i \to j$, add it in.

Ex.





etc (finish yourself)

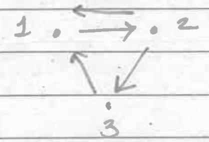
Let's do this with adjacency matrices

A in above example = $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

Take $(A + A^2 + A^3 + \cdots + A^n)$ → tells us all possible connections in G.

→ change all positive entries to 1

Eg.



$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$(A + A^2 + A^3) = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

adjacency matrix of transitive closure

** Transitive closure, another way

(via Boolean matrix operations)

Boolean operations review

"addition" and "multiplication" on $\{0, 1\}$ ("FALSE", "TRUE")

We have operations

- $\vee$ = ("OR") = Boolean addition
- $\wedge$ = ("AND") = Boolean multiplication

Def: $a \vee b = \begin{cases} 1 & \text{if at least one of } a, b \text{ is } 1 \\ 0 & \text{otherwise} \end{cases}$

$a \wedge b = \begin{cases} 1 & \text{if both } a, b \text{ equal } 1 \\ 0 & \text{otherwise} \end{cases}$