

* Last time: Transitive closure of R
 = smallest superset R' of R (i.e. $R \subseteq R'$) such that
 R' is transitive.

Adjacency matrix of transitive closure:

If $A = \text{adj. matrix of a graph with } n \text{ vertices}$,
 take $(A + A^2 + \dots + A^n) = B$, and change B so that

- all non-zero (positive) entries become 1
- all zero-entries kept as-is.

The changed B is the adj. matrix of the transitive closure.

* Today: Do the same w/ Boolean arithmetic

Operations $\left. \begin{array}{l} \vee = \text{OR} \\ \wedge = \text{AND} \end{array} \right\} \text{ binary operations on } \{0, 1\}$
"FALSE" "TRUE"

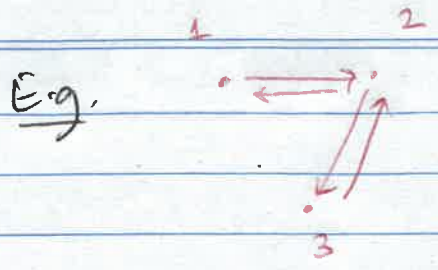
$$\begin{array}{l} 0 \vee 1 = 1 \vee 0 = 1 \vee 1 = 1 \\ 0 \vee 0 = 0 \end{array}$$

$$\begin{array}{l} 1 \wedge 1 = 1 \\ 1 \wedge 0 = 0 \wedge 1 = 0 \wedge 0 = 0 \end{array}$$

(Like an addition)

(Like a multiplication)

* Boolean matrix product = ~~as~~ like usual matrix product,
 but replace every instance of $+$ with \vee and
 \times with \wedge .



$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

A^{*2} = 2nd Boolean power
 (* = Boolean matrix product)

$$A * A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$(A * A)_{(2,2)}$ = Boolean dot product of

$[1, 0, 1]$ with $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$= (1 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 1) = 1 \vee 0 \vee 1 = 1$$

↑ replaces x

↑ replaces +

Theorem / consequence : Let A be the adj matrix of a graph with n vertices.
 The adjacency matrix of the transitive closure of A is :

$$(A \vee A^{*2} \vee A^{*3} \vee \dots \vee A^{*n})$$

* Aside: Given a relation R , you can define the

- ① reflexive closure
- ② symmetric closure

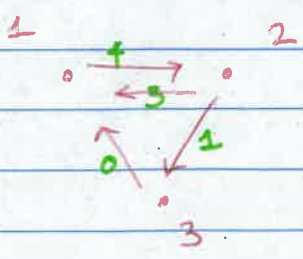
(Remember, the $\langle \text{blah} \rangle$ closure of R is the smallest relation R' such that $R \subseteq R'$ and R' is $\langle \text{blah} \rangle$)

Q: What about anti-symmetric closure?
(This doesn't make sense \rightarrow can you figure out why?)

Q: ~~Q~~ How to modify the ~~edge~~ adjacency matrix to get the adj matrix of the reflexive/symmetric closure?

* Weighted graphs

A weighted graph is some $G = (V, E)$, together with a weight (a number) for each edge.



(Think of weights as either a cost or a capacity for each edge.)

\rightarrow Can construct the weighted adjacency matrix

$$W = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} * & 4 & * \\ 3 & * & 1 \\ 0 & * & * \end{pmatrix} \end{matrix}$$

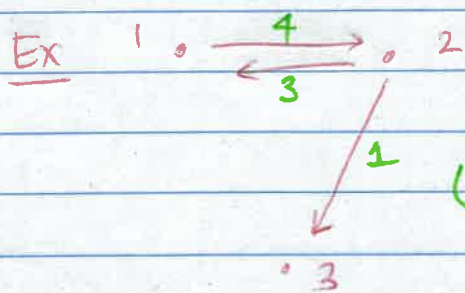
$(i, j)^{th}$ entry = weight of the edge from i to j
 * if no edge $i \rightarrow j$

What should \star be?
→ Depends on the application.

For us, we focus on regarding the weights as costs.

If there is no edge from i to $j \Rightarrow$ "the cost is too high to pay"

→ we'll use ∞ to denote this, except at $i=j$
 $\star \infty$ is not a number, it is a symbol, we'll see how to handle it in calculations.



Find shortest-cost (cheapest) path from i to j

$$W = \begin{bmatrix} 0 & 4 & \infty \\ 3 & 0 & 1 \\ \infty & \infty & 0 \end{bmatrix}$$