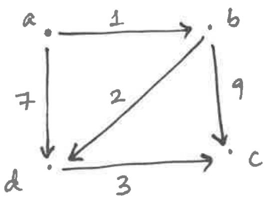
MATH 2301 14/08/2023

* Admin : See announcement about quizzes.

* Last time : Weighted adjacency matrices



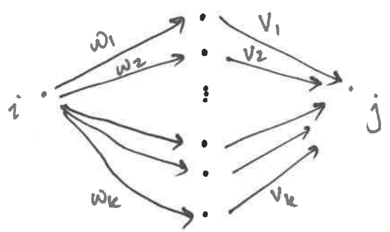
$$W = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & \infty & \cancel{7} \\ \infty & 0 & 9 & 2 \\ \infty & \infty & 0 & \infty \\ \infty & \infty & 3 & 0 \end{bmatrix} \end{array}$$

0 on diagonal
∞ when no edge $\Big\}$ our conventions for our application.

We want : least cost of travelling from i to j.

W tells us the least costs of travelling from i to j in 0 or 1 steps

→ From this, compute for $\boxed{\le 2}$ steps ?



possible costs:

$w_1 + v_1$
$w_2 + v_2$
$\vdots$
$w_k + v_k$ $\Big\}$ depending on your route.

$\boxed{\text{we want: } \min\{w_1+v_1, w_2+v_2, \dots, w_k+v_k\}}$

Note : some "edges" can actually be weight $= 0$ connections, which could include staying in the same spot, hence total length ≤ 2.

We use the "min-plus" dot product.

addition $+$ is replaced by min
multiplication $\times$ is replaced by $+$

as follows:

Let w and v be vectors. (same length)

$$\boxed{w \odot v := \min\{w_1+v_1, w_2+v_2, \dots, w_k+v_k\}}$$

$\odot$ denotes min-plus vector/matrix product.

* How to deal with "∞" ?

Suppose ~~w~~ c is a number (ie $c \ne \infty$)

$$\boxed{\begin{aligned} c+\infty &:= \infty + c := \infty \\ \infty + \infty &:= \infty \end{aligned}}$$

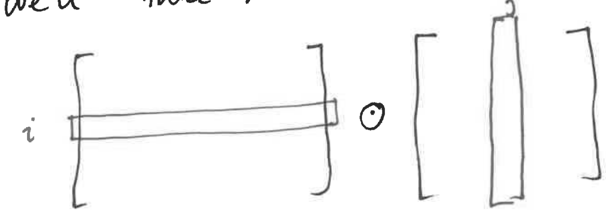
$$\boxed{\begin{aligned} \min\{c, \infty\} &:= c \\ \min\{\infty, \infty\} &:= \infty \end{aligned}}$$

REMEMBER : "∞" is a placeholder symbol to denote a number bigger than anything else we encounter.

* Matrix min-plus product

Let A & B be weighted adjacency matrices.
Typically we'll take $A = B = W$ (our wtd adj matrix)

$A \odot B =$



$(i,j)^{th}$ entry $=$ min-plus dot product of R_i & C_j
$= R_i \odot C_j$

So if $A = B = W$

$W \odot W = W^{\odot 2}$, and more generally, take

$$\boxed{W^{\odot n} = n^{th} \text{ min-plus matrix power}}$$

* **Example**

$$\begin{bmatrix} 0 & 1 & \infty & 7 \\ \infty & 0 & 9 & 2 \\ \infty & \infty & 0 & \infty \\ \infty & \infty & 3 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & \infty & 7 \\ \infty & 0 & 9 & 2 \\ \infty & \infty & 0 & \infty \\ \infty & \infty & 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{0} & & & \boxed{3} \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

$\min\{0+0,\ 1+\infty,\ \infty+\infty,\ 7+\infty\}$
$= \min\{0, \infty, \infty, \infty\}$
$= 0$

$\min\{0+7,\ 1+2,\ \infty+\infty,\ 7+0\}$
$= \min\{7, 3, \infty, 7\} = 3$

> Q: (Aside): How to compute max costs?
> → See worksheet

* **Consequence**: The $(i,j)^{th}$ entry of $W \odot W$ is the min. cost of going from i to j in ≤ 2 steps

* **Theorem**: The $(i,j)^{th}$ entry of $W^{\odot n}$ is the min. cost of going from i to j in $\leq n$ steps.

* **Remarks**:

- For the purposes of computing min-costs, we will restrict ourselves to weights ≥ 0.
- Usually we don't have self-loops of positive weight; even if we do, we'll take the cost of going from i to i as 0.

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* **Recall**: If G has n vertices then the shortest path from i to j (if it exists) has length $\leq n$.
In fact if $i \neq j$, it has length $\leq (n-1)$.

Suppose $m \geq n$. Then:
$W^{\odot m}$ records min-costs of paths of length $\leq m$

* Since we have non-negative weights, shorter paths must have lesser (or equal) weights.

* **Theorem**: Let G have n vertices.
Suppose $m \geq n$. Then,
$W^{\odot m} = W^{\odot (n-1)}$.

Explanation: If $i = j$ then the min cost from $i \to j$ is zero, already achieved in the length $= 0$ path.
If $i \neq j$, the min-cost from $i \to j$ is already achieved by a path of length $\leq n-1$.
Beyond $(n-1)$, you don't add any new info.

* Partial order relations.

Def: Let S be a set; R a relation. on S.
We say that R is a total order if:

for any two $a, b \in S$: either
$(a, b) \in R$ or
$(b, a) \in R$ or ~~both~~
$a = b$

and exactly one of these holds.

R is transitive

Another way to define this:
R is a total order if
① R is reflexive
② R is anti-symmetric
③ R is transitive.
④ For each $a, b \in$ ~~R~~ S, either $(a,b) \in R$ or $(b,a) \in R$.