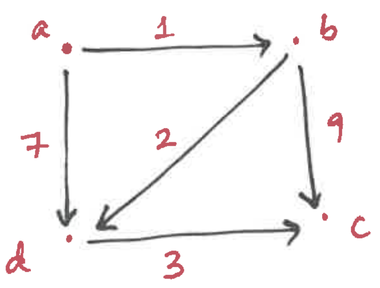
MATH 2301 14/08/2023 

* Admin : See announcement about quizzes.

* Last time : Weighted adjacency matrices



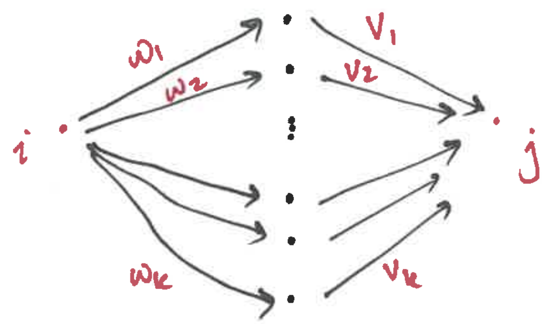

$$W = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & \infty & 7 \\ \infty & 0 & 9 & 2 \\ \infty & \infty & 0 & \infty \\ \infty & \infty & 3 & 0 \end{bmatrix} \end{array}$$

$\left.\begin{array}{l} 0 \text{ on diagonal} \\ \infty \text{ when no edge} \end{array}\right\}$ our conventions for our application.

We want : least cost of travelling from $i$ to $j$.

$W$ tells us the least costs of travelling from $i$ to $j$ in $0$ or $1$ steps

→ From this, compute for $\boxed{\leq 2}$ steps ?




possible costs:

$\left.\begin{array}{l} w_1 + v_1 \\ w_2 + v_2 \\ \vdots \\ w_k + v_k \end{array}\right\}$ depending on your route.

$\boxed{\text{we want: } \min\{w_1+v_1, w_2+v_2, \ldots, w_k+v_k\}}$

Note : Some "edges" can actually be weight $= 0$ connections, which could include staying in the same spot, hence total length $\leq 2$.

We use the "<u>min-plus" dot product</u>.

addition $+$ is replaced by min

multiplication $\times$ is replaced by $+$

as follows:

Let $w$ and $v$ be vectors. (same length)

$$\boxed{w \odot v := \min\{w_1+v_1,\ w_2+v_2,\ \dots,\ w_k+v_k\}}$$

denotes min-plus vector/matrix product.

* How to deal with "$\infty$" ?

Suppose $c$ is a number (ie $c \neq \infty$)

$$\boxed{\begin{aligned} c+\infty &:= \infty + c := \infty \\ \infty + \infty &:= \infty \end{aligned}}$$

$$\boxed{\begin{aligned} \min\{c, \infty\} &:= c \\ \min\{\infty, \infty\} &:= \infty \end{aligned}}$$

REMEMBER: "$\infty$" is a placeholder symbol to denote a number bigger than anything else we encounter.

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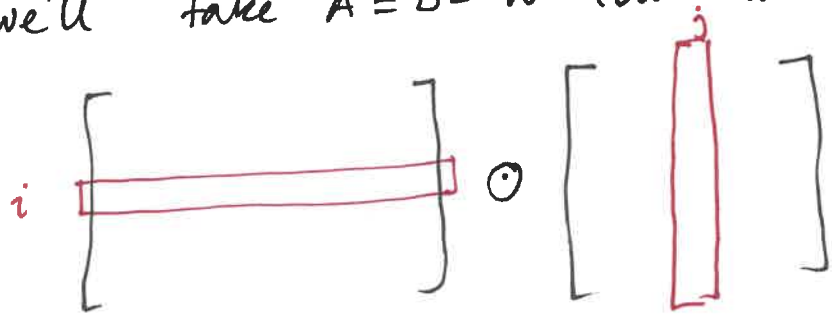
* <u>Matrix min-plus product</u>

Let $A$ & $B$ be weighted adjacency matrices.

Typically we'll take $A = B = W$ (our wtd adj matrix)

$A \odot B =$



$(i,j)^{th}$ entry $=$ min-plus dot product of $R_i$ & $C_j$

$= R_i \odot C_j$

So if $A = B = W$

$W \odot W = W^{\odot 2}$, and more generally, take

$$\boxed{W^{\odot n} = n^{th} \text{ min-plus matrix power}}$$

* Example

$$\begin{bmatrix} 0 & 1 & \infty & 7 \\ \infty & 0 & 9 & 2 \\ \infty & \infty & 0 & \infty \\ \infty & \infty & 3 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & \infty & 7 \\ \infty & 0 & 9 & 2 \\ \infty & \infty & 0 & \infty \\ \infty & \infty & 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & & & 3 \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

$\min\{0+0,\ 1+\infty,\ \infty+\infty,\ 7+\infty\}$
$= \min\{0, \infty, \infty, \infty\}$
$= 0$

$\min\{0+7,\ 1+2,\ \infty+\infty,\ 7+0\}$
$= \min\{7, 3, \infty, 7\} = 3$

Q: (Aside): How to compute max costs?
→ See worksheet

* Consequence: The $(i,j)^{th}$ entry of $W \odot W$ is the min. cost of going from $i$ to $j$ in $\leq 2$ steps

* Theorem: The $(i,j)^{th}$ entry of $W^{\odot n}$ is the min. cost of going from $i$ to $j$ in $\leq n$ steps.

* Remarks :

- For the purposes of computing min-costs, we will restrict ourselves to weights $\geq 0$.
- Usually we don't have self-loops of positive weight; even if we do, we'll take the cost of going from $i$ to $i$ as $0$.

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* Recall : If $G$ has $n$ vertices then the shortest path from $i$ to $j$ (if it exists) has length $\leq n$.
In fact if $i \neq j$, it has length $\leq (n-1)$.

Suppose $m \geq n$. Then:
$W^{\odot m}$ records min-costs of paths of length $\leq m$

* Since we have non-negative weights, shorter paths must have lesser (or equal) weights.

* Theorem : Let $G$ have $n$ vertices.
Suppose $m \geq n$. Then,
$$W^{\odot m} = W^{\odot (n-1)}.$$

Explanation : If $i = j$ then the min cost from $i \to j$ is zero, already achieved in the length $= 0$ path.
If $i \neq j$, the min-cost from $i \to j$ is already achieved ~~too~~ by a path of length $\leq n-1$.
Beyond $(n-1)$, you don't add any new info.

## * Partial order relations.

Def: Let $S$ be a set; $R$ a relation. on $S$.
We say that $R$ is a total order if:

for any two $a, b \in S$: either
$(a, b) \in R$ or
$(b, a) \in R$ or ~~both~~
$a = b$

and exactly one of these holds.

$R$ is transitive

Another way to define this:
$R$ is a total order if

1. $R$ is reflexive
2. $R$ is anti-symmetric
3. $R$ is transitive.
4. For each $a, b \in \cancel{R}\, S$, either $(a,b) \in R$ or $(b,a) \in R$.