MATH 2301 16/08/2023 ①

* Last time:

Def: A total order/total ordering on a set S is a relation R that is:

(1) Reflexive,

(2) Anti-symmetric, and

(3) Transitive, such that

(4) for any $a, b \in S$ we have $(a,b) \in R$ or $(b,a) \in R$.

Example: $S = \mathbb{N} = \{0, 1, 2, 3, \cdots\}$ with $R = \{(a,b) \mid a \leq b\}$

(Exercise: check the properties.)

Notation: We usually say $a \leq_R b$ if $(a,b) \in R$ for a total order.

* Today:

Def: A partial order/partial ordering on a set S is a relation R on S that is:

(1) Reflexive

(2) ~~Symmetric~~ Anti-symmetric, and

(3) Transitive.

* Notation: If R is a partial order relation, we'll usually write $a \preceq_R b$ for $(a,b) \in R$.

* (Obvious) examples: $\mathbb{N}$ with $R = \{(a,b) \mid a \leq b\}$

Any total order is also a partial order.

②

* Examples

(1) Let $S = \mathcal{P}(A)$ (power set), where A is (say) $\{a, b, c\}$

Subset relation: We'll say that ~~$X \subseteq$~~

$X \preceq Y$ for $X, Y \in \mathcal{P}(A)$ (i.e, $X, Y \subseteq A$) if $X \subseteq Y$.

① Reflexivity: yes, because $X \subseteq X$.

② Anti-symmetry: if $X \subseteq Y$ and $X \neq Y$, then $Y \not\subseteq X$; so yes.

③ Transitivity: if ~~$X \subseteq Y$~~ $X \subseteq Y$ and $Y \subseteq Z$ then $X \subseteq Z$; so yes

Note: $X = \{a, b\}$ and $Y = \{b, c\}$ but $X \not\subseteq Y$ and $Y \not\subseteq X$

so it is not a total order.

(2) Let $S = \{1, 2, 3, 4, 5, 6\}$. Divisibility relation.

$R = \{(a,b) \mid a \text{ is a factor of } b\}$

i.e. b is divisible by a

Notation: if a is a factor of b, we say "a divides b", we write $a \mid b$

$R = \{(1,1), (2,2), \cdots, (6,6),$
$(1,2), (1,3), (1,4), (1,5), (1,6),$
$(2,4), (2,6), (3,6)\}$

* Exercise: Check that it is a partial order relation.

Note: $(2,3) \notin R$ and $(3,2) \notin R$

$(4,6) \notin R$ and $(6,4) \notin R$.

$\Rightarrow$ it is not a total order.

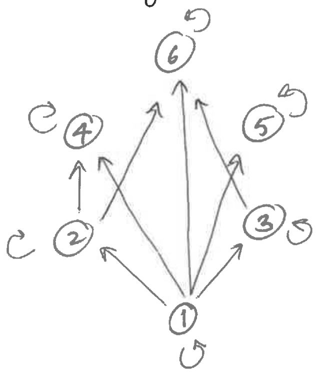
* Hasse diagrams

Example : $S = \{1, \dots, 6\}$ + divisibility relation

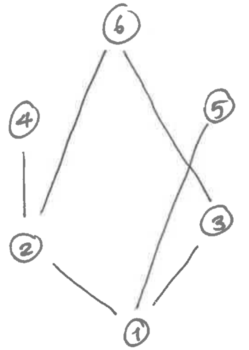
Notational aside : If S has a partial order relation $\preceq$ on it, then $(S, \preceq)$ is called a poset [= partially ordered set]

S is our poset; it has divisibility relation.

A Hasse diagram is a cleaned-up version of the graph of our partial order relation.



(Graph)



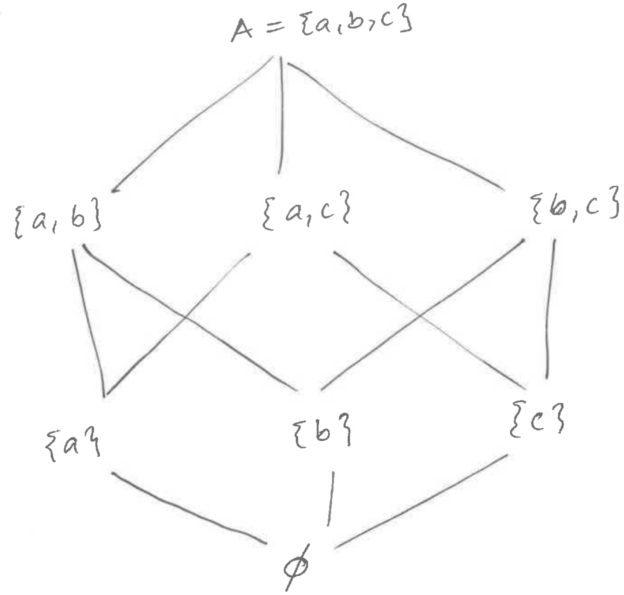
(Hasse diagram)

Construction : Given the graph of a partial order relation, we can draw a Hasse diagram as follows:

1. Re-arrange so that all arrows go upwards.
2. Delete all loops (self-loops)
3. Delete all edges that are a consequence of transitivity :

4. Delete arrow-heads.

$A = \{a, b, c\}$, $S = \mathcal{P}(A)$, subset relation.



[Aside : Why does this look like a cube??]

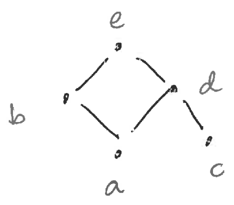
Note : Given a Hasse diagram, you can reconstruct the graph (reverse steps from previous page)

"Abstract Hasse diagrams"

(abstract representations of posets)

* Maximum, minimum, maximal, minimal elements

Consider a poset with the following Hasse diagram:



Note: $(c,e) \in R$ or $c \preceq e$

$c \npreceq b$ and $b \npreceq c$

(no upward oriented path)

$(a,e) \in R$

$x \preceq y$ if there is a path from x to y that goes / flows upwards

Observations:

* e is bigger than everything else.
* a and c are "bottommost" but <u>not</u> smaller than everything else.

$a \npreceq c$, $c \npreceq a$, $c \npreceq b$.