MATH 2301

* **Last time:**

Def: A total order / total ordering on a set S is a relation R that is:

(1) Reflexive,
(2) Anti-symmetric, and
(3) Transitive, such that
(4) for any $a, b \in S$ we have $(a,b) \in R$ or $(b,a) \in R$.

Example: $S = \mathbb{N} = \{0, 1, 2, 3, \cdots\}$ with $R = \{(a,b) \mid a \leq b\}$

(Exercise: check the properties.)

Notation: We usually say $a \leq_R b$ if $(a,b) \in R$ for a total order.

* **Today:**

Def: A partial order / partial ordering on a set S is a relation R on S that is:

(1) Reflexive
(2) ~~Symmetric~~ Anti-symmetric, and
(3) Transitive.

* **Notation**: If R is a partial order relation, we'll usually write $a \preceq_R b$ for $(a,b) \in R$.

* **(Obvious) examples**: $\mathbb{N}$ with $R = \{(a,b) \mid a \leq b\}$

Any total order is also a partial order.

* Examples

(1) Let $S = P(A)$, where A is (say) $\{a, b, c\}$ power set

Subset relation : We'll say that

$X \preceq Y$ for $X, Y \in P(A)$ if $X \subseteq Y$.
(i.e, $X, Y \subseteq A$)

1. Reflexivity : yes, because $X \subseteq X$.
2. Anti-symmetry : if $X \subseteq Y$ and $X \neq Y$, then $Y \nsubseteq X$; so yes.
3. Transitivity : if $X \subseteq Y$ and $Y \subseteq Z$ then $X \subseteq Z$; so yes

Note : $X = \{a, b\}$ and $Y = \{b, c\}$ but $X \nsubseteq Y$ and $Y \nsubseteq X$

so it is not a total order.

(2) Let $S = \{1, 2, 3, 4, 5, 6\}$. Divisibility relation.
$R = \{(a, b) \mid a$ is a factor of $b\}$
i.e. b is divisible by a

Notation: if a is a factor of b, we say "a divides b", we write $a \mid b$

$$R = \{(1,1), (2,2), \cdots, (6,6), (1,2), (1,3), (1,4), (1,5), (1,6), (2,4), (2,6), (3,6)\}$$

* Exercise : Check that it is a partial order relation.

Note : $(2,3) \notin R$ and $(3,2) \notin R$
$(4,6) \notin R$ and $(6,4) \notin R$.

$\Rightarrow$ it is not a total order.

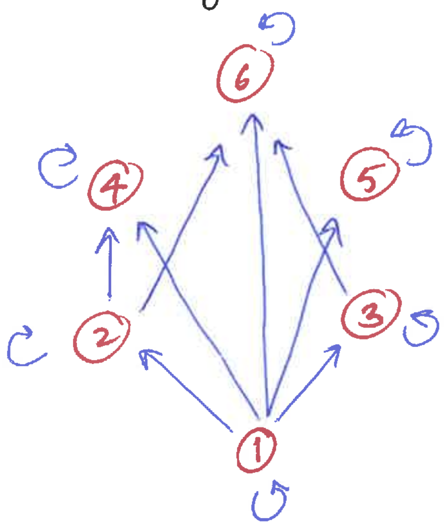
* Hasse diagrams

Example : $S = \{1, \dots, 6\}$ + divisibility relation

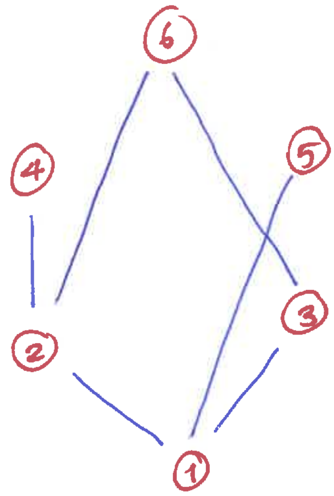
Notational aside : If S has a partial order relation $\preceq$ on it, then $(S, \preceq)$ is called a poset [= partially ordered set]

S is our poset; it has divisibility relation.

A Hasse diagram is a cleaned-up version of the graph of our partial order relation.



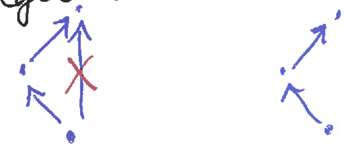
(Graph)



(Hasse diagram)

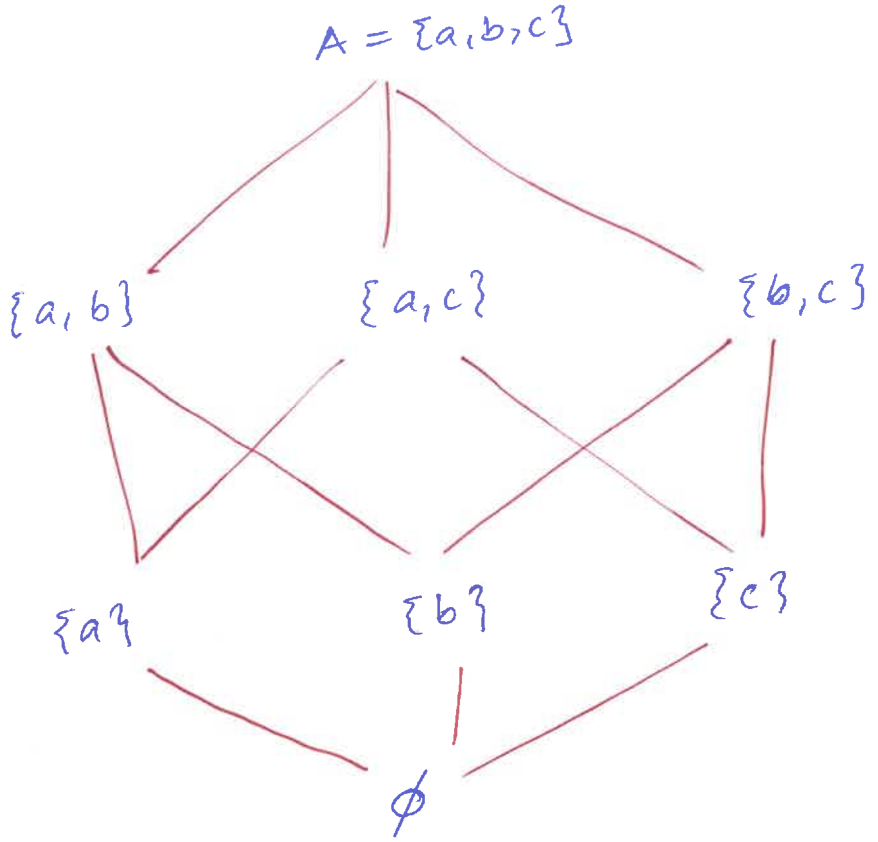
Construction : Given the graph of a partial order relation, we can draw a Hasse diagram as follows:

1. Re-arrange so that all arrows go upwards.
2. Delete all loops (self-loops)
3. Delete all edges that are a consequence of transitivity :



4. Delete arrow-heads.

$A = \{a, b, c\}$, $S = \mathcal{P}(A)$, subset relation.



[Aside: Why does this look like a cube??]

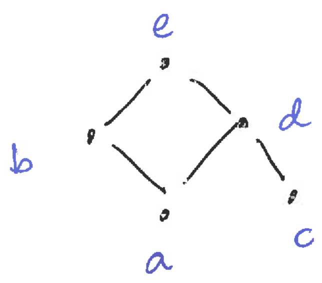
Note: Given a Hasse diagram, you can reconstruct the graph (reverse steps from previous page)

"Abstract Hasse diagrams"

(abstract representations of posets)

* Maximum, minimum, maximal, minimal elements

Consider a poset with the following Hasse diagram:



<u>Note</u>: $(c,e) \in R$ or $c \preceq e$

$c \npreceq b$ and $b \npreceq c$

(no upward oriented path)

$(a,e) \in R$

$x \preceq y$ if there is a path from x to y that goes / flows upwards

<u>Observations</u>:

* e is bigger than everything else.
* a and c are "bottommost" but <u>not</u> smaller than everything else.

$a \npreceq c$, $c \npreceq a$, $c \npreceq b$.