MATH 2301 21/08/2023

* Admin : Class rep class survey. (see announcement)

* Last time : Topological sorting → Today : implications

Example :

Original ordering : (a, b, c, d)
(not topological)

Topological sorting example : ~~[illegible]~~
(b, a, d, c)

Adjacency matrices :

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & & & & \\ b & & & & \\ c & & & & \\ d & & & & \end{array}\quad \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Can have nonzero entries ~~above~~ below main diagonal.

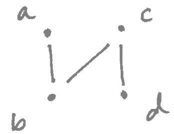
$$\begin{array}{c} \\ b \\ a \\ d \\ c \end{array}\begin{array}{c} \begin{array}{cccc} b & a & d & c \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

All zeroes below main diagonal.

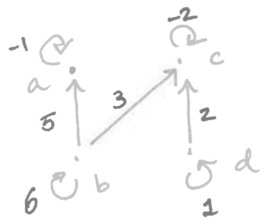
Prop : (Consequence of def) : If $(P, \preceq)$ is a poset and $(v_1, \dots, v_n)$ a topological sorting of P then the adjacency matrix in this ordering is upper-triangular, i.e. all entries below the main diagonal are zero.

* Algebra on posets. (Incidence algebra).

Example :



Defn : An edge function f on a poset $(P, \preceq)$ is a weight associated to every edge in the poset relation.
That is, for each $x \preceq y$ in P, we have a number $f([x, y])$



(previous example as weighted graph)

We get an edge function, say f on P, which sends $[x, y]$ (whenever $x \preceq y$) to a number as shown in the figure.

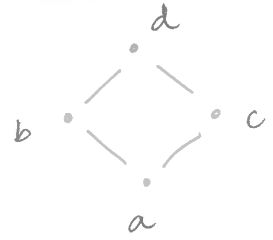
E.g. $f([d, d]) = 1$
$f([b, c]) = 3$
$f([c, c]) = -2$
etc.

* If $x \npreceq y$, we conventionally set $f([x, y]) = 0$.

* We will soon be interested in the set of all edge functions on a poset $(P, \preceq)$.
We'll call this set $\mathcal{A}(P)$. (Note : it is an infinite set)

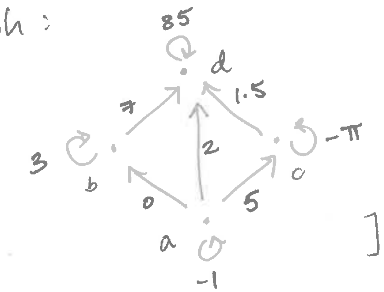
③

E.g.



← P.

[Graph :



]

(+ edge function f)

To specify an edge function, we separately choose a number for <u>every</u> edge.

* We'll leverage (variant) weighted adjacency matrices to specify edge functions.

Given a poset $(P, \preceq)$ and an edge function f we can write the corresponding matrix M_f, for example in the above situation:

$$M_f = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} -1 & 0 & 5 & 2 \\ 0 & 3 & 0 & 7 \\ 0 & 0 & -\pi & 1.5 \\ 0 & 0 & 0 & 85 \end{bmatrix} \end{array}$$

← <u>Rmk</u>: Note that (a,b,c,d) is already a topological sorting, so this matrix M_f and any such matrix M_g is upper-triangular. But it is not a requirement of the construction.

[It is, however, convenient]

④

* <u>Example</u> (from previous pictures on page 2)

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} -1 & \boxed{0} & \boxed{0} & \boxed{0} \\ 5 & 6 & 3 & \boxed{0} \\ \boxed{0} & \boxed{0} & -2 & \boxed{0} \\ \boxed{0} & \boxed{0} & 2 & 1 \end{bmatrix} \end{array}$$

← these zeroes are forced by the structure of $(P, \preceq)$

<u>Note</u>: Giving ~~the~~ the matrix M_f on $(P, \preceq)$ is the same amount of data as giving f directly.

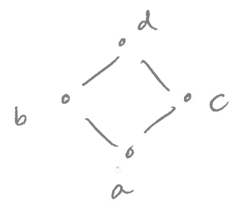
<u>Recall</u>: We are interested in the set of all edge functions f on a given poset $(P, \preceq)$. (Called $\mathcal{A}(P)$).

Equivalently, we look at all $n \times n$ matrices where the non-zero entries only appear at the spots where adjacency matrix is non-zero.

<u>Def</u> (Notation): Let $(P, \preceq)$ be a poset. Let $x, y \in P$. Then the interval (closed interval) $[x,y]$ is the following set:

$$[x,y] = \{ z \in P \mid x \preceq z \text{ and } z \preceq y \}$$

<u>Example</u>:



$[a,d] = \{a, b, c, d\}$.