23 Aug 2023

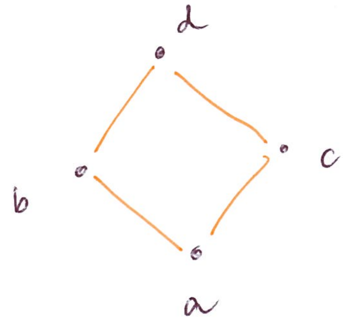
MATH 2301

* Last time:

Let $(P, \preceq)$ be a poset. Then the (closed) interval $[x, y]$ for $x, y \in P$ is the set

$$[x,y] = \{z \in P \mid x \preceq z \preceq y\}.$$

E.g.



$[a,d] = \{a, b, c, d\}$

$[c,d] = \{c, d\}$

$[b,c] = \emptyset$

$[d,d] = \{d\}$

* Def: Let $(P, \preceq)$ be a poset. Then $I(P)$ is the set of all non-empty intervals of P:

$$I(P) = \{[x,y] \mid x \preceq y\}$$

* Def: An edge function f on a poset P is a function $f : I(P) \to \mathbb{R}$

real numbers

Example:

$$I(P) = \{\underbrace{[a,a]}_{=\{a\}}, \underbrace{[a,b]}_{=\{a,b\}}, \underbrace{[b,b]}_{=\{b\}}\}$$

An edge function is an assignment sending any $[x,y]$ in $I(P)$ to a real number; e.g.

f is: $[a,a] \mapsto 3$, $[a,b] \mapsto -25$, $[b,b] \mapsto 2\pi$

$$M_f = \begin{array}{cc} a \\ b \\ \end{array} \begin{bmatrix} 3 & -25 \\ 0 & 2\pi \end{bmatrix}$$
$$\qquad\quad a \qquad b$$

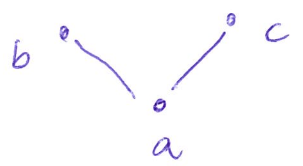
[There are no restrictions on what numbers you can choose for each non-empty interval]

* Some special edge functions (Fix a poset $(P, \preceq)$)

- The zeta function ζ is an edge function, defined as follows:

$$\zeta([x,y]) = 1 \quad \text{whenever } x \preceq y.$$

E.g.:



$$M_\zeta = \begin{array}{c|ccc} & & & \\ a & 1 & 1 & 1 \\ b & 0 & 1 & 0 \\ c & 0 & 0 & 1 \\ \hline & a & b & c \end{array}$$

Note: M_ζ is always the adjacency matrix of the poset!

- The delta function δ is defined as follows:

$$\delta([x,y]) = \begin{cases} 1 & \text{whenever } x = y \\ 0 & \text{otherwise} \end{cases}$$

$$M_\delta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: This is the 3×3 identity matrix. M_δ will always be the identity matrix of the appropriate size, for any poset P.

* The incidence algebra (Fix a poset $(P, \preceq)$)

Notation: The set of all edge functions on P is written as $\mathcal{A}(P)$

In particular, ζ and δ are elements of $\mathcal{A}(P)$.

Let's look at operations on $\mathcal{A}(P)$ [algebraic operations]

** Addition

Let $f, g \in \mathcal{A}(P)$, with matrices M_f, M_g.

Then $(f+g) \in \mathcal{A}(P)$, defined as:

$$(f+g)([x,y]) := f([x,y]) + g([x,y])$$ ← this is an eqn of real numbers

Note :

$M_{(f+g)}$ = matrix of $(f+g)$

$M_{(f+g)} = M_f + M_g$ → Check if not clear!

** Scalar multiplication

If $f \in \mathcal{A}(P)$ with associated matrix M_f and $c \in \mathbb{R}$, then cf is the edge function defined as:

$$(cf)([x,y]) = c \cdot f([x,y])$$

$$M_{(cf)} = \underbrace{c \cdot M_f}$$ ← multiply every entry of M_f by c.

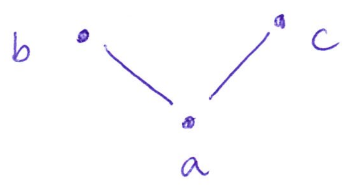
[Using a combination of + and scalar product, I can take linear combinations, e.g. $5f - 17g$, etc.]

** Matrix product

Let $f, g \in \mathcal{A}(P)$ be edge functions with matrices M_f and M_g.

One could take $(M_f \cdot M_g)$ = matrix product, to get another $(n \times n)$ matrix.

Example :



$$M_f = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 2 & 3 & 5 \\ \boxed{0} & -1 & \boxed{0} \\ \boxed{0} & \boxed{0} & 7 \end{bmatrix} \end{array}$$

The boxed entries are preset to be zero.

$$M_g = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & -1 & 3 \\ \boxed{0} & -2 & \boxed{0} \\ \boxed{0} & \boxed{0} & 2 \end{bmatrix} \end{array}$$

In this case,

$$(M_f \cdot M_g) = \begin{bmatrix} 2 & 3 & 5 \\ 0 & -1 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -8 & 16 \\ \boxed{0} & 2 & \boxed{0} \\ \boxed{0} & \boxed{0} & 14 \end{bmatrix}$$

Q: If we take $(M_f \cdot M_g)$ and then look at an entry corresponding to $[x, y]$ where $x \npreceq y$, do we automatically always get 0?

If yes, then $(M_f \cdot M_g)$ will be the matrix of another edge function, otherwise it won't be!

Let's see.

Let $(P, \preceq)$ be a poset, M_f and M_g as before (matrices of edge fns.)

Let us compute

$$(M_f \cdot M_g)_{(x,y)} = \sum_{z \in P} (M_f)_{(x,z)} \cdot (M_g)_{(z,y)}$$

(dot product expansion)