MATH 2301 25/08/2023 ①

* Last time : Operations on $\mathcal{A}(P)$ = set of edge functions on P, also called the incidence algebra.

* Given $f \in \mathcal{A}(P)$, we have associated matrix M_f.

Conversely, consider any matrix M such that $M_{x,y} = 0$ whenever $x \npreceq y$.

Then there is $f \in \mathcal{A}(P)$ such that $M = M_f$.

$f([x,y]) = (x,y)^{th}$ entry of M

Note : If $x \npreceq y$, $M_{(x,y)} = 0$, so there is no extra info in the matrix M.

* We covered addition & scalar multiplication for edge functions.

* Matrix product of M_f and M_g.

Let $(P, \preceq)$ be a poset; f and $g \in \mathcal{A}(P)$

$$\boxed{(M_f \cdot M_g)_{(x,y)} = \sum_{z \in P} (M_f)_{(x,z)} \cdot (M_g)_{(z,y)}} \quad \text{— ①}$$

Note : If $z \in P$ such that $x \npreceq z$, then $(M_f)_{(x,z)} = 0$. Similarly, if $z \npreceq y$, then $(M_g)_{(z,y)} = 0$.

In the sum above, it is enough to restrict to z such that $x \preceq z \preceq y$; i.e, $z \in [x,y]$.

$$\Rightarrow \boxed{(M_f \cdot M_g)_{(x,y)} = \sum_{z \in [x,y]} (M_f)_{(x,z)} \cdot (M_g)_{z,y}} \quad \text{— ②}$$

②

* Proposition : $(M_f \cdot M_g)_{(x,y)} = 0$ if $x \npreceq y$.

Reason: Look at formula above. If $x \npreceq y$, then $[x,y] = \emptyset$. Then RHS of formula is zero, because it is an empty summation.

Upshot : $(M_f \cdot M_g)$ is ALSO the matrix of an edge function!

* Example : see last time's notes.

* Def : Let $f, g \in \mathcal{A}(P)$ for a fixed poset $(P, \preceq)$. Then the convolution product of f and g is a new element of $\mathcal{A}(P)$, which is the function corresponding to the matrix $(M_f \cdot M_g)$.

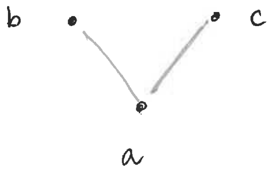
We call this new function $(f * g)$.

$$(f*g)([x,y]) := (M_f \cdot M_g)_{(x,y)}$$

* Restating the previous matrix formula ②, we get

$$\boxed{(f*g)([x,y]) = \sum_{z \in [x,y]} f([x,z]) \cdot g([z,y])} \quad \text{— ③}$$

* Examples:



$\zeta([x,y]) = 1 \quad \text{if } x \preceq y$

$\delta([x,y]) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$

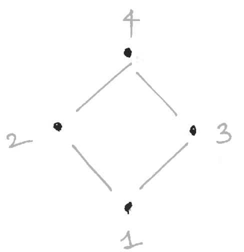
- $(\zeta * \zeta)([a,c]) = \sum_{z \in [a,c]} \underbrace{\zeta([a,z])}_{=1} \cdot \underbrace{\zeta([z,c])}_{=1}$

$= \zeta([a,a]) \cdot \zeta([a,c]) + \zeta([a,c]) \cdot \zeta([c,c])$

$= 1 \cdot 1 + 1 \cdot 1 = \boxed{2}$

Recall: M_ζ = adjacency matrix

(in this case, $(\zeta * \zeta)([a,c])$ = # paths of length 2 from a to c in the graph of the relation.)



$f([x,y]) = x + y.$

$f([1,4]) = 5$

$f([2,3]) =$ UNDEFINED, technically.

b/c $[2,3) = \emptyset$

But $(M_f)_{(2,3)} = 0$.

$(\zeta * f)([1,4]) = \sum_{z \in [1,4]} \zeta([1,z]) \cdot f([z,4])$

$\parallel$

$(M_\zeta \cdot M_f)_{(1,4)}$

$= \zeta([1,1]) \cdot f([1,4]) + \zeta([1,2]) f([2,4])$

$+ \zeta([1,3]) \cdot f([3,4]) + \zeta([1,4]) f([4,4])$

- Let $(P, \preceq)$ any poset. $f \in \mathcal{A}(P)$

$\delta \in \mathcal{A}(P)$ as before.

Look at $(f * \delta)_{(x,y)} = (M_f \cdot \underbrace{M_\delta}_{= I})_{(x,y)}$

I identity matrix.

$= (M_f \cdot I)_{(x,y)} = (M_f)_{(x,y)}$

So, $(f * \delta) = f$!

Similarly, $(\delta * f) = f$!

$\Rightarrow$ $\delta \in \mathcal{A}(P)$ is the multiplicative identity for the convolution product.