

* Last time: Operations on $\mathcal{A}(P)$ = set of edge functions on P , also called the incidence algebra.

* Given $f \in \mathcal{A}(P)$, we have associated matrix M_f .

Conversely, consider any matrix M such that

$$M_{x,y} = 0 \text{ whenever } x \not\leq y.$$

Then there is $f \in \mathcal{A}(P)$ such that $M = M_f$.

$$f([x,y]) = (x,y)^{\text{th}} \text{ entry of } M$$

Note: If $x \not\leq y$, $M_{(x,y)} = 0$, so there is no extra info in the matrix M .

* We covered addition & scalar multiplication for edge functions.

* Matrix product of M_f and M_g .

Let (P, \leq) be a poset; f and $g \in \mathcal{A}(P)$.

$$(M_f \cdot M_g)_{(x,y)} = \sum_{z \in P} (M_f)_{(x,z)} \cdot (M_g)_{(z,y)} \quad - \quad (1)$$

Note: If $z \in P$ such that $x \not\leq z$, then $(M_f)_{(x,z)} = 0$.

Similarly, if $z \not\leq y$, then $(M_g)_{(z,y)} = 0$.

In the sum above, it is enough to restrict to z such that $x \leq z \leq y$; i.e., $z \in [x,y]$.

$$\Rightarrow (M_f \cdot M_g)_{(x,y)} = \sum_{z \in [x,y]} (M_f)_{(x,z)} \cdot (M_g)_{(z,y)} \quad - \quad (2)$$

* Proposition: $(M_f \cdot M_g)_{(x,y)} = 0$ if $x \not\leq y$.

Reason: Look at formula above. If $x \not\leq y$, then $[x,y] = \emptyset$.
Then RHS of formula is zero, because it is an empty summation.

Upshot: $(M_f \cdot M_g)$ is ALSO the matrix of an edge function!

* Example: see last time's notes.

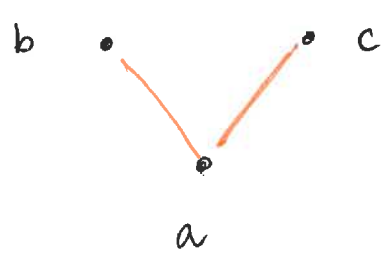
* Def: Let $f, g \in \mathcal{A}(P)$ for a fixed poset (P, \leq) .
Then the convolution product of f and g is a new element of $\mathcal{A}(P)$, which is the function corresponding to the matrix $(M_f \cdot M_g)$.
We call this new function $(f * g)$.

$$(f * g)([x, y]) := (M_f \cdot M_g)_{(x, y)}$$

* Restating the previous matrix formula. ②, we get

$$(f * g)([x, y]) = \sum_{z \in [x, y]} f([x, z]) \cdot g([z, y]) \quad \text{③}$$

* Examples:



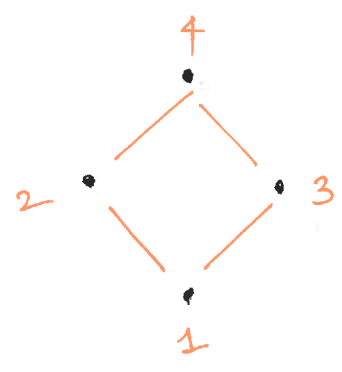
$$\zeta([x,y]) = 1 \quad \text{if } x \leq y$$

$$\delta([x,y]) = \begin{cases} 1 & \text{if } x=y \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} - (\zeta * \zeta)([a,c]) &= \sum_{z \in [a,c]} \underbrace{\zeta([a,z])}_1 \cdot \underbrace{\zeta([z,c])}_1 \\ &= \zeta([a,a]) \cdot \zeta([a,c]) + \zeta([a,c]) \cdot \zeta([c,c]) \\ &= 1 \cdot 1 + 1 \cdot 1 = \boxed{2} \end{aligned}$$

Recall: $M_\zeta =$ adjacency matrix

(in this case, $(\zeta * \zeta)([a,c]) =$ # paths of length 2 from a to c in the graph of the relation.)



$$f([x,y]) = x+y.$$

$$f([1,4]) = 5$$

$$f([2,3]) = \text{UNDEFINED, technically.} \\ \text{b/c } [2,3] = \emptyset$$

$$\text{But } (M_f)_{(2,3)} = 0.$$

$$\begin{aligned} (\zeta * f)([1,4]) &= \sum_{z \in [1,4]} \zeta([1,z]) \cdot f([z,4]) \\ &= \zeta([1,1]) \cdot f([1,4]) + \zeta([1,2]) \cdot f([2,4]) \\ &\quad + \zeta([1,3]) \cdot f([3,4]) + \zeta([1,4]) \cdot f([4,4]) \end{aligned}$$

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$$(M_\zeta \cdot M_f)_{(1,4)}$$

- Let (P, \leq) any poset. $f \in \mathcal{A}(P)$
 $\delta \in \mathcal{A}(P)$ as before.

$$\begin{aligned} \text{Look at } (f * \delta)_{(x,y)} &= (M_f \cdot \underbrace{M_\delta}_{\substack{= \\ \mathbf{I} \text{ identity matrix.}}})_{(x,y)} \\ &= (M_f \cdot \mathbf{I})_{(x,y)} = (M_f)_{(x,y)} \end{aligned}$$

$$\text{So, } (f * \delta) = f !$$

$$\text{Similarly, } (\delta * f) = f !$$

$\Rightarrow \delta \in \mathcal{A}(P)$ is the multiplicative identity for the convolution product.