MATH 2301 28/08/2023 ①

* Admin: Teaching break next 2 weeks!
HW 6 (longer) due on 22 Sept (in 3 weeks).
Quiz solutions coming soon.

* Last time: Let $(P, \preceq)$ be a poset.
$\delta \in \mathcal{A}(P)$ is the multiplicative identity for the convolution product, i.e. if $f \in \mathcal{A}(P)$ then

$$(f * \delta) = (\delta * f) = f.$$

[Reason: $M_\delta = I$, and $M_f \cdot I = M_f = I \cdot M_f$]

* Today: Invertibility and the Möbius function.

Def: We say that $f \in \mathcal{A}(P)$ is invertible if: there is some $g \in \mathcal{A}(P)$ such that

$$\boxed{f * g = \delta}$$

Prop: If there is some $g \in \mathcal{A}(P)$ such that $f * g = \delta$, then actually $g * f = \delta$ and vice-versa. Also, the inverse function g is unique.

Q: How to find inverses?? → Later.

(See HW for more): The equation $f * g = \delta$ means that for every $x \preceq y$, we have

$$(f*g)([x,y]) = \delta([x,y]) \quad \leftarrow \text{either } 1 \text{ or } 0.$$

Plug in $y = x$ to get $(f*g)([x,x]) = \delta([x,x]) = 1$.

②

Prop: An edge function $f \in \mathcal{A}(P)$ is invertible if and only if $f([x,x]) \neq 0$ for every $x \in P$.

* Example

Consider f such that

$$M_f = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} -3 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix} \end{array}$$

Q: Is f invertible?

A: NO, because $f([b,b]) = 0$. (There is a zero on the main diagonal.)

* Corollary: If $f \in \mathcal{A}(P)$ and M_f has no zeroes on the main diagonal, then there is an inverse function $g \in \mathcal{A}(P)$.

* Reminder: $M_{(f*g)} = M_f \cdot M_g$.

if $f * g = \delta$, then:

$$(M_f \cdot M_g) = M_{f*g} = M_\delta = I.$$

i.e. $\boxed{M_f \cdot M_g = I}$ → Given M_f, we can use linear algebra to compute M_g.

Similarly, $g * f = \delta$ gives $\boxed{M_g \cdot M_f = I}$

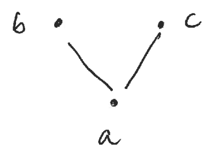
We won't assume we know how to do this.

* Notation: If there is some $g \in \mathcal{A}(P)$ such that $f * g = \delta$, then g is called the inverse of f, and denoted f^{-1}. [not the same as in calculus!!]

* Möbius functions

Recall $\zeta \in \mathcal{A}(P)$ is the edge function such that $\zeta([x,y]) = 1$ for $x \preceq y$.

E.g.



$$M_\zeta = \begin{array}{c|ccc} & a & b & c \\ \hline a & 1 & 1 & 1 \\ b & 0 & 1 & 0 \\ c & 0 & 0 & 1 \end{array}$$

(adjacency matrix)

no zeroes!

Abstractly, we see that ζ is invertible.

Def: Let $(P, \preceq)$ be a poset. Then the Möbius function $\mu \in \mathcal{A}(P)$ is the unique function such that $\boxed{\zeta * \mu = \mu * \zeta = \delta}$. That is, $\mu := \zeta^{-1}$.

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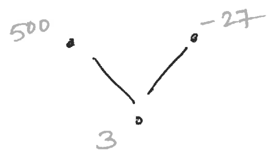
* Note: We don't yet know how to calculate μ in practice.

We'll come back to μ.

E.g.

Def: A vertex function $p : P \to \mathbb{R}$ is an assignment of a real number to each vertex in P

$p =$



i.e. we choose a number for each vertex.

$$v_p = \begin{array}{c} a \\ b \\ c \end{array}\begin{bmatrix} 3 \\ 500 \\ -27 \end{bmatrix}$$

We can represent a vertex function p by a vector v_p, where the i^{th} entry of v_p is the value of p on the i^{th} vertex.

* Note: Fix poset $(P, \preceq)$

We have edge functions $f : I(P) \to \mathbb{R}$ → matrix M_f

vertex functions $p : P \to \mathbb{R}$ → vectors v_p.

Given an edge function f and a vertex fn p, we can consider

$$\underbrace{M_f}_{n \times n} \cdot \underbrace{v_p}_{n \times 1}$$

← output is another $n \times 1$ matrix, i.e. another vector, i.e. can be regarded as a vertex function.

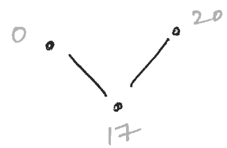
E.g.

$$M_f = \begin{bmatrix} -3 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$v_p = \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}$$

$$M_f \cdot v_p = \begin{bmatrix} -3 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 9+4+4 \\ 0 \\ 20 \end{bmatrix} = \begin{bmatrix} 17 \\ 0 \\ 20 \end{bmatrix}$$

Output can be viewed as another vertex function:



So, given $f \in \mathcal{A}(P)$ and $p : P \to \mathbb{R}$, we can produce another vertex function via $M_f \cdot v_p$.

We'll call the new function $\boxed{f * p}$ ← different from usual convolution...

* <u>One-sided convolution</u>

Given $f \in \mathcal{A}(P)$ and $p : P \to \mathbb{R}$, define

$(f * p)$ = one-sided convolution of f with p, to be the vertex function whose vector is

$(M_f \cdot v_p)$.

We can also define $p * f$ to be the vertex function whose row vector is $\cancel{(v_p^t * M_f)}$

$$(v_p^t \cdot M_f)$$
