# MATH 2301

30/08/2023 

* **Admin**: Teaching break next week + week after.

* **Last time**: One-sided convolution.

Let $(P, \leq)$ be a poset.

Let $f \in A(P)$ be an edge fn and $p : P \to \mathbb{R}$ a vertex function.

Then $\nu_p$ = column vector of $p$ = $\begin{bmatrix} p(1) \\ p(2) \\ \vdots \\ p(n) \end{bmatrix}$ if vertices $\{1, 2, \ldots, n\}$

(Row vector of $p$ would be $\nu_p^t = [p(1) \; p(2) \; \ldots \; p(n)]$)

**Def**: The (left) one-sided convolution $f * p$ is the vertex function whose vector is

$$(\underbrace{M_f}_{n\times n} \cdot \underbrace{\nu_p}_{n\times 1})$$ ← $n \times 1$ vector, i.e. a column vector.

**Def**: The (right) one-sided convolution $p * f$ is the vertex function whose row vector is

$$(\underbrace{\nu_p^t}_{1\times n} \cdot \underbrace{M_f}_{n\times n})$$ ← $1 \times n$ vector, i.e. a row vector

## * Example


Consider $v_p = \begin{array}{c} (a) \\ (b) \\ (c) \end{array} \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$ column vector of some vertex fn.

Take $f = \zeta$ ; $M_f = M_\zeta = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then $M_f \cdot v_p = M_\zeta \cdot v_p$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ -1 \end{bmatrix}$$

By def, this is the column vector of a new vertex fn on the poset, and we call that new fn $(f * p) = (\zeta * p)$.

Similarly,

$$v_p^t \cdot M_\zeta = [3 \;\; 4 \;\; -1] \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [3 \;\; 7 \;\; 2]$$

is the row vector of $(p * \zeta)$, by def.

* We can see that

$$(M_f \cdot v_p)_x =$$

$$\sum_y (M_f)_{(x,y)} \cdot (v_p)_y$$

matrix product expansion.

$$= \sum_{x \preceq y} (M_f)_{(x,y)} \cdot (v_p)_y$$

other entries of $M_f$ are zero.

* $(v_p^t \cdot M_f)_x =$

$$= \sum_z (v_p^t)_z (M_f)_{(z,x)}$$

$$= \sum_{z \preceq x} (v_p^t)_z (M_f)_{(z,x)}$$

similar to above.

[see next page]

* $$\boxed{(M_f \cdot \nu_p)_x = \sum_{x \preceq y} (M_f)_{(x,y)} \cdot (\nu_p)_y.}$$

Translating to the def of $(f*p)$, we see

$$\boxed{(f*p)(x) = \sum_{x \preceq y} f([x,y]) \cdot p(y)}$$

* Similarly:

$$\boxed{(\nu_p^t \cdot M_f)_x = \sum_{z \preceq x} (\nu_p)_z^t \cdot (M_f)_{(z,x)}}$$

Translating to the def. of $p*f$, we see:

$$\boxed{(p*f)(x) = \sum_{z \preceq x} p(z) \cdot f([z,x])}$$

[Remember: The outputs $f*p$ and $p*f$ are vertex functions.]

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* Reminder: $\zeta \in \mathcal{A}(P)$ is an invertible edge function.

Def: $\mu \in \mathcal{A}(P)$ is the inverse function to $\zeta$, i.e.

$\mu$ = Möbius function.

$$\boxed{\mu * \zeta = \delta = \zeta * \mu}$$

* Möbius inversion [Fix $(P, \preceq)$ a poset.]

Suppose that $p, q : P \to \mathbb{R}$ are vertex functions such that

$(\zeta * q) = p$.

E.g.


$v_q = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$ ;

$\boxed{p = \zeta * q}$ has vector $v_p = \begin{bmatrix} 6 \\ 4 \\ -1 \end{bmatrix}$

(see page 2; q is the previous p)

(1) $\boxed{M_\zeta \cdot v_q = v_p}$ ← Consider this eqn and recall that $\mu * \zeta = \delta$

Multiply (1) on the left by $M_\mu$:

(2) $\boxed{M_\mu \cdot M_\zeta \cdot v_q = M_\mu \cdot v_p}$

Since $M_\mu \cdot M_\zeta = I$, we get

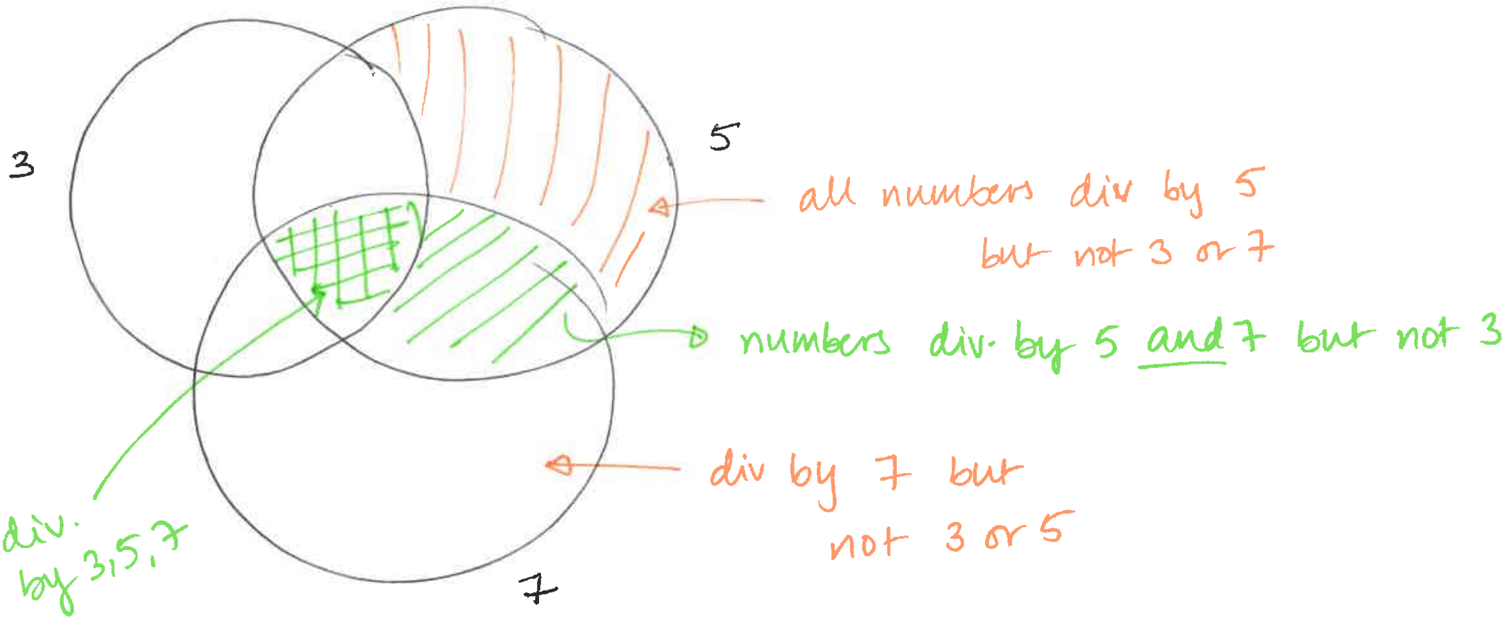
(3) $\boxed{v_q = M_\mu \cdot v_p}$.

* In the situation that $\boxed{p = \zeta * q}$, we can find a formula for $q$ as:

$\boxed{q = \mu * p}$ ← Möbius inverse formula.

# Application : The inclusion-exclusion principle.

E.g. Question : How many numbers from 1 to 2023 are not divisible by 3, 5, or 7 ?




Answer = 2023 − #(numbers in ~~one of~~ the regions above)

We'll use posets instead.

Note : Number of integers in [1, 2023] that are :

- divisible by 3 is $\left[\frac{2023}{3}\right] = 674$ ← integer part

- div. by 5 is $\left[\frac{2023}{5}\right] = 404$

- div. by 7 is $\left[\frac{2023}{7}\right] = 289$

[Finish on Friday ...]