MATH 2301 1/09/2023

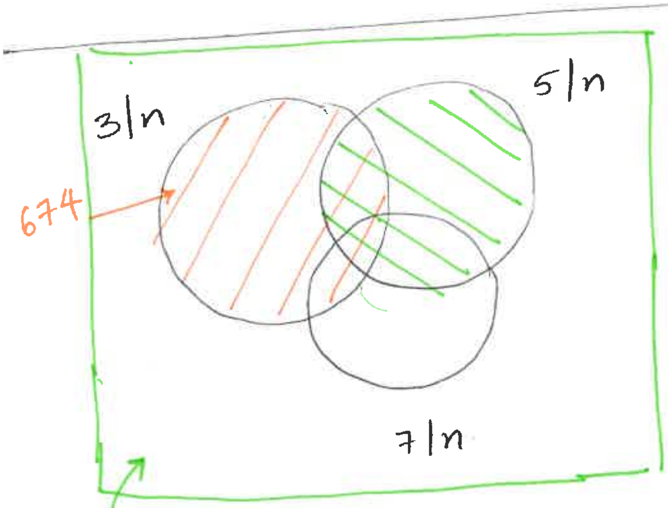
* Last time: Q: How many integers in $[1, 2023]$ are not divisible by $3, 5,$ or 7?

We counted:

$\#$ ints divisible by $3 = \left\lfloor \frac{2023}{3} \right\rfloor = 674$ (integer part)

$\#$ ints divisible by $5 = \left\lfloor \frac{2023}{5} \right\rfloor = 404$

$\#$ ints divisible by $7 = \left\lfloor \frac{2023}{7} \right\rfloor = 289$



How many outside all circles?

~~Need~~

Claim: To find $\#$ ints not divisible by any of them, we need to find the $\#$ ints divisible by at least one of them & subtract this number from 2023

BUT: If we take $674 + 404 + 289$, that is too many!

Note: The circles overlap, so this has some double-counting!

* Need to add /subtract pairwise, triplewise intersections

- # ints divisible by 3 and 5 : $\Bigg\}$ = # ints divisible by 15 $= \left\lfloor \frac{2023}{15} \right\rfloor = 134$

- # ints divisible by 3 and 7 : $\Bigg\}$ = # ints divisible by 21 $\cancel{\left\lfloor \frac{2023}{25} \right\rfloor}\ \left\lfloor \frac{2023}{21} \right\rfloor = 96.$

- # ints divisible by 5 and 7 $\Bigg\}$ = # ints divisible by 35 $= \left\lfloor \frac{2023}{35} \right\rfloor = 57$

- # ints divisible by 3, 5, and 7 : $\Bigg\}$ = # ints divisible by 105 $= \left\lfloor \frac{2023}{105} \right\rfloor = 19.$

(triple intersection)

Reminder on principle of inclusion/exclusion:
ints divisible by at least one of 3, 5, 7 is

$= \sum(\text{single intersections}) - \sum(\text{double intersections}) + \sum(\text{triple intersections})$ ---

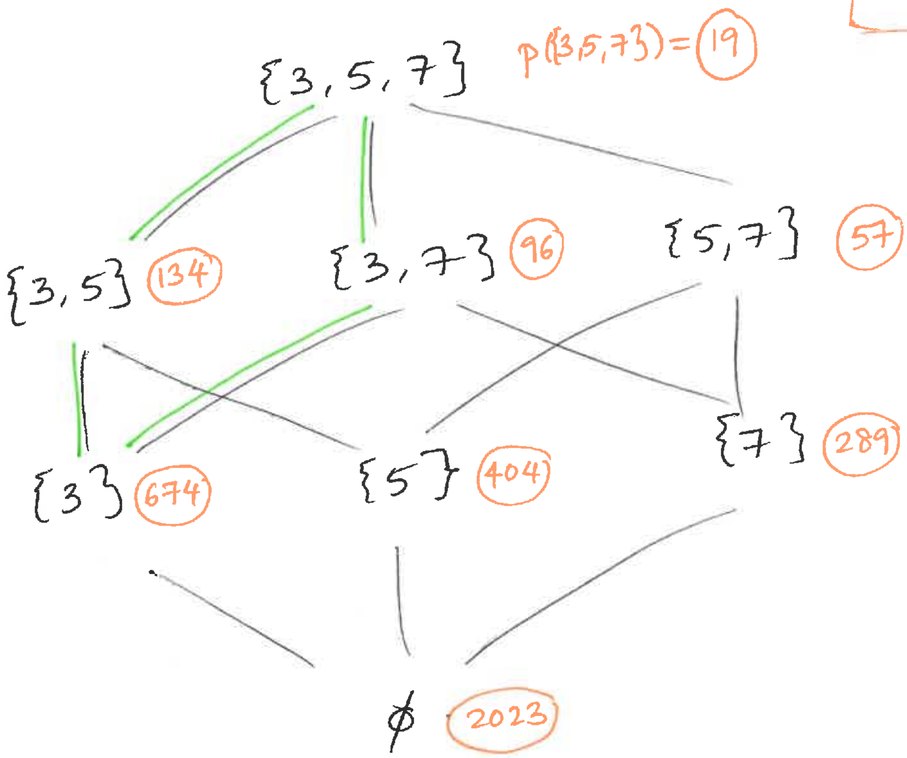
$= [674+404+289] - [134+96+57] + [19]$

* Consider subset poset of $S = \{3, 5, 7\}$.

$P = \mathcal{P}(S)$

(We care about $\{1, 2, \dots, 2023\}$ aren't divisible by anything in S)

circled = p-values



For $A \subseteq S$, let $p(A)$ = # integers in $[1, 2023]$ that are divisible by everything in A

(see orange circled values)

Let $q(A)$ = # integers in $[1, 2023]$ that are divisible by everything in A and not divisible by anything in $S \setminus A$.

Note : We want # elts not divisible by anything in S

i.e. we want to compute $q(\phi)$.

* Note :

$$\boxed{p(A) = \sum_{A \subseteq B} q(B)}$$

E.g. $A = \{3\}$

$q(A)$ includes things div. by only 3, not 5, not 7

$q(\{3,5\})$ = things div. by only 3 & 5, not 7

$q(\{3,7\})$ = div by 3 & 7, not 5

$q(\{3,5,7\})$ = div by 3, 5, and 7

Their sum = everything div by 3

(In example):

$$\boxed{p(\{3\}) = q(\{3\}) + q(\{3,5\}) + q(\{3,7\}) + q(\{3,5,7\})}$$

The previous formula translates to:

$$\boxed{p = \zeta * q}$$

(Check using convolution formula)

We want the value of q in terms of p:

Use Möbius inversion:

$$\boxed{q = \mu * p}$$

In particular, $q(\phi) = (\mu * p)(\phi)$.

Formula for μ?

* Formulas for μ.

① If P is any poset, then:

(a) $\mu([x,x]) = 1$

(b) if $x \precneqq y$ then

$$\boxed{\mu([x,y]) = -\sum_{z \in [x,y)} \mu([x,z])}$$

$z \in [x,y)$ means $x \preceq z \precneqq y$.

② If P is a subset poset, then:

(a) $\mu([A,A]) = 1$

(b) If $A \subseteq B$, then

$$\mu([A,B]) = (-1)^{|B \setminus A|}$$

elts in B that are not in A.

Use this to compute $q(\phi)$ in previous example.

$\boxed{q = \mu * p}$, $\quad q(\phi) = \sum_{\phi \subseteq A} \mu([\phi, A]) \cdot p(A)$

$$q(\phi) = \sum_{\phi \subseteq A} (-1)^{|A|} \cdot p(A)$$

$$q(\phi) = 2023 - (674 + 404 + 289) + (134 + 96 + 57) - 19$$

$$\boxed{q(\phi) = 924}$$

There are 924 integers in [1, 2023] that are not divisible by 3, 5, or 7.