MATH 2301 11/10/2023 ①

* Last time: Discussed the existence of non-regular languages.

* Specific example: $L = \{0^n 1^n \mid n \geq 0\}$.

Claim: this is not regular.

If it were regular, there would be a DFA M with N states (for some N) that accepted exactly the words in L.

Recall: if $|w| > N$, then the path of w through M would repeat a state

i.e. $w = x\,y\,z$

↳ starts & ends at a repeated state; $y \neq \varepsilon$.

~~Note: w ends on an accep~~

Suppose w is accepted. Then it ends on an accept state. This means that all of the following strings are accepted: xz, xyz, $xyyz$, $xyyyz$, $xyyyyyz$, etc.

$L = \{0^n 1^n\}$; M a DFA with N states.

$\boxed{w = 0^{N+1} 1^{N+1}}$; w accepted by M

By previous analysis, we know: w repeats a state within the first N letters; $\boxed{w = x\,y\,z}$

↳ begins & ends at the same state

$y \neq \varepsilon$ and $|xy| \leq N$

Note: for $w = 0^{N+1} 1^{N+1}$, the xy portion only has 0s.

So y is a concatenation of 0s; say $y = 0^k$

But then we also accept the strings xz, $xyyz$, etc.

$w = 0^{N+1} 1^{N+1}$; $y = 0^k$, $x = \cancel{0^{N+1-k}}\ 0^{\ell}$

$z = 0^{N+1-(k+\ell)} 1^{N+1}$

and $k \neq 0$

$xz = 0^{\ell} 0^{N+1-(k+\ell)} 1^{N+1} = 0^{(N+1)-k} 1^{N+1}$ ← M has to accept this.

$L = \{0^n 1^n \mid n \geq 0\}$ ⟺ so $xz = 0^{(N+1)-k} 1^{N+1}$ is not in L.

$\Rightarrow$ M is wrong; i.e. $L(M) \neq L$.

Consequence: There is No DFA which accepts exactly the language L!

⟺ L is not regular.

Rmks: There is a theorem, called the pumping lemma for regular languages, which encodes the above argument.

We will not formally state it.

You will not be expected to write such proofs on the exam.

Games (Impartial combinatorial games)

What is a game (in our context)?

- Two players (P1 and P2)
- Game state at each turn
- Turn-based
- Impartial : the possible set of next moves depends only on the game state, not on whose turn it is
  (E.g. chess is not impartial.)
- Has perfect information (no secrets/ hidden information)
- Not random in any way
- After each game state, there are only finitely many reachable game states, and there are no directed cycles.
  (can't go back to previous state)
- A player wins if the next player has no moves left to make.

Example: Subtraction game with $S = \{1, 3, 4\}$

Start state (or any game state) is a non-negative integer.

e.g 15

Allowed moves: subtract a single element of S, reaching another non-negative number.

Let's play! $S = \{1, 3, 4\}$

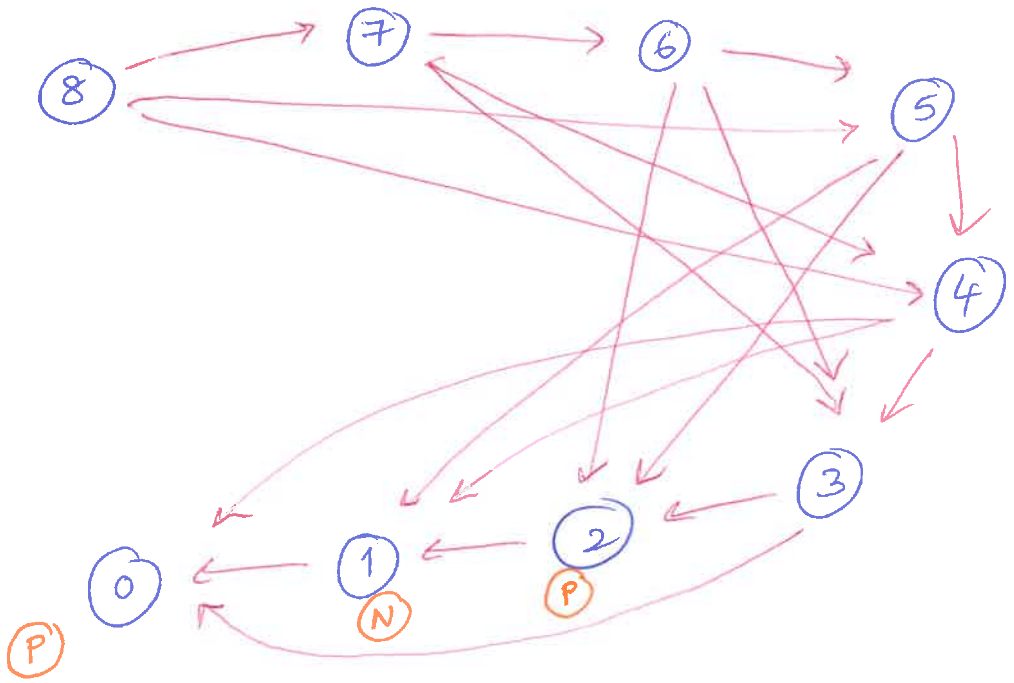
$15 \to 11 \to 8 \to 5 \to 2 \to 1 \to 0$

$21 \to 17 \to 14 \to 13 \to 9 \to 5 \to 2 \to 1 \to 0$

Q: Who will win? How to win?

Game graph.

$n = 8$



P label $\leftrightarrow$ "previous player ~~[illegible]~~ wins"

N label $\leftrightarrow$ "next player wins"

[More on Friday]