MATH 2301

16/10/2023

* **Last time:** N and P labelling, hackenbush & chomp. + mirroring (for chomp)

* **Today:** Nim.

Game states: some number of piles of smarties.

Allowed moves: choose one pile and eat one or more smarties from that pile.

Example

3 6 8

$\swarrow$ $\searrow$ $\rightarrow$ ...

(1,6,8) (3,3,8)

$\downarrow$

(1,8) $\leftarrow$ write as (8,1) if you prefer
(states $\leftrightarrow$ multi-sets, i.e. sets with repetition)

* Easy case : If there is only one non-empty pile, it is an N-state.

→ Eat the whole pile to leave 0.

* Slightly harder : If there are two equal piles :

→ Mirror what the first player does:

E.g. $(5,5) \to (4,5) \to (4,4)$ etc

So, states of the form (a,a) are P states.

* Slightly harder : If there are two non-empty unequal piles (a,b) with $a > b > 0$:

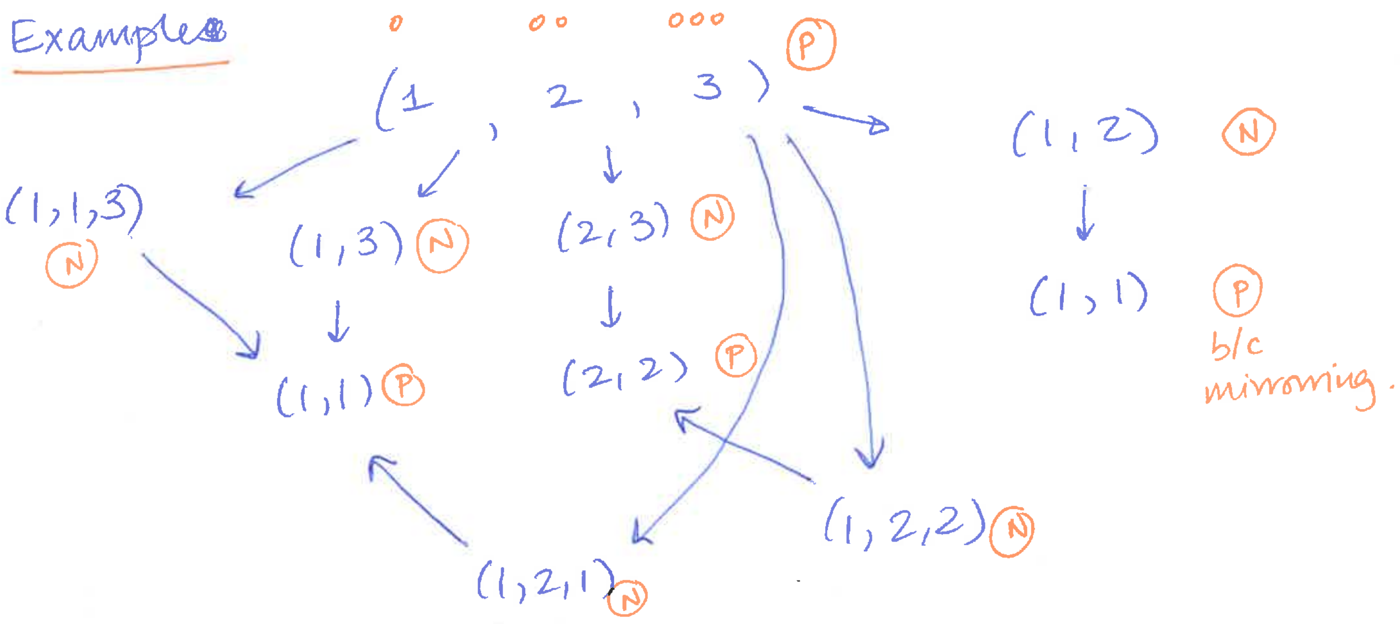
→ First player can equalise the piles to make (b,b); which is a P-state.

So, states of this form are N-states.

* Harder : Three or more non-empty piles ...

Our task will be to solve this fully (without the need to draw any game graphs)

* Examples



[PARTIAL game graph]

⊛ By working backwards from terminal states (states w/o outgoing arrows,) it is possible to label every state as N/P.

* Strategy : Mirroring, levelled up!

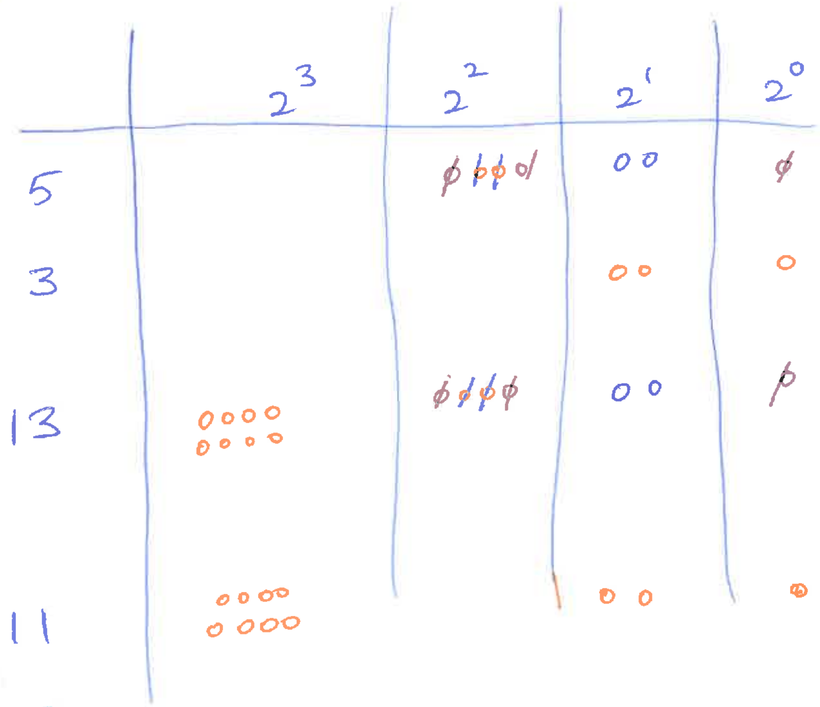
Let us look at (3, 5, 6)

Break up the piles into columns by binary representation, as follows:

	.... 2^2	2^1	2^0
3		o o	o
5	o o o o		o
6	o o o o	o o	

Note: Each column has an even number of sub-piles.

Another example:



This corresponds to the binary representations

$$
\begin{array}{rrrrrr}
 & 5 = & & 1 & 0 & 1_2 \\
 & 3 = & & & 1 & 1_2 \\
 & 13 = & 1 & 1 & 0 & 1_2 \\
\oplus & 11 = & \oplus\ 1 & 0 & 1 & 1_2 \\
\hline
 & & 0 & 0 & 0 & 0_2
\end{array}
$$

Take the XOR. (exclusive or)

$1 \oplus 1 = 0$
$0 \oplus 0 = 0$
$1 \oplus 0 = 1$
$0 \oplus 1 = 1$

These 4 numbers XOR to 0 $\equiv$ the columns above have an even # of entries each.

Def: The nim-sum of non-negative integers $(m_1, m_2, \dots, m_k)$ is the integer whose binary representation is the XOR of the binary representations of each of the m_i.

E.g. $(1, 2, 3)$

$$
\begin{array}{rcr}
 & 1 & = & 0\,1_2 \\
 & 2 & = & 1\,0_2 \\
\oplus & 3 & = & 1\,1_2 \\
\hline
 & 0 & & 0\,0_2
\end{array}
$$

XOR each column.

Nim-sum of $(1, 2, 3)$ is: $1 \oplus 2 \oplus 3 = 0$.

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* Theorem (to be proved): A game state $(m_1, m_2, \dots, m_k)$ of nim is an N-state if and only if $m_1 \oplus \cdots \oplus m_k > 0$.

It is a P-state if and only if

$$m_1 \oplus \cdots \oplus m_k = 0.$$