MATH 2301 18/10/2023 (1)

* Last time : nim

Example $(3, 5, 10)$ → $(3, 1, 10)$ → $(5, 10)$ etc

(example moves)

* 2 piles of equal size = P position.

* Last time : Saw that you write each pile size in binary, take XOR.

$$\begin{array}{rcr}
3 & = & 11_2 \\
5 & = & 101_2 \\
\oplus\ 10 & = & \oplus\ 1010_2 \\
\hline
12 & & 1100_2
\end{array}$$

XOR:

$1 \oplus 1 = 0 \oplus 0 = 0$

$1 \oplus 0 = 0 \oplus 1 = 1$

"exclusive OR"

nim-sum

Theorem : Let $(m_1, \dots, m_k)$ be a nim position.

It is an N-position iff $m_1 \oplus \dots \oplus m_k > 0$

It is a P-position iff $m_1 \oplus \dots \oplus m_k = 0$.

Note : The nim-sum of $(m_1, \dots, m_k)$ is zero iff each column in the binary representations has an even number of 1s.

(2)

E.g.

$$\begin{array}{rcr}
3 & = & 11_2 \\
5 & & 101_2 \\
\oplus\ 10 & & \oplus\ 1010_2 \\
\hline
12 & & 1100_2
\end{array}$$

vs

$$\begin{array}{rcr}
3 & = & 11_2 \\
5 & = & 101_2 \\
\oplus\ 6 & = & 110_2 \\
\hline
0 & & 000_2
\end{array}$$

Notice : To make nim-sum 0, you need to cancel ~~the 1~~ (in particular) the leftmost 1 from the nim-sum

E.g.

$$\begin{array}{rcr}
10 & = & \boxed{1010_2} \\
13 & = & 1101_2 \\
12 & = & 1100_2 \\
\oplus\ 8 & = & \cancel{1}000_2 \\
\hline
3 & = & \boxed{0011_2}
\end{array}$$

← make a move

Pf sketch of theorem:

$(m_1, \cdots, m_k)$ is some nim-position.

Case 1: Suppose that $m_1 \oplus \cdots \oplus m_k = 0$.

Properties of the nim-sum

(1) $x \oplus x = 0$

(2) If $x \oplus y = z$ then

$$x \oplus y \oplus y = z \oplus y$$

i.e. $x = z \oplus y = y \oplus z$

E.g. $(1, 2, 3)$

1	$=$	01_2
2	$=$	10_2
$\oplus\ 3$	$=$	11_2
		00_2

If $(m_1, \cdots, m_k)$ is such that $m_1 \oplus \cdots \oplus m_k = 0$ then, any move results in a non-zero nim-sum.

Pf: If you make a move in the first pile (say) $m_1 \longrightarrow m_1'$ (by eating some smarties)

New nim-sum is:

$$(m_1' \oplus m_2 \oplus m_3 \oplus \cdots \oplus m_k) = s$$

$$s = s \oplus \underbrace{(m_1 \oplus m_2 \oplus \cdots \oplus m_k)}_{= 0}$$

$$s = (m_1' \oplus \cancel{m_2} \oplus \cdots \oplus \cancel{m_k}) \oplus (m_1 \oplus \cancel{\cdots} \oplus \cancel{m_k})$$

$$s = m_1' \oplus m_1$$

Note: $x \oplus y = 0$ iff $x = y$.

So, since $m_1' \neq m_1$, we see $s \neq 0$.

$\Rightarrow$ Case 1 done: If $(m_1 \oplus \cdots \oplus m_k) = 0$, then any move makes the nim-sum > 0.

Case 2: $(m_1, \cdots, m_k)$ a position such that $m_1 \oplus \cdots \oplus m_k > 0$.

E.g.

	3	$=$	11_2
	5	$=$	101_2
$m_1 =$	$\oplus\ 10$	$=$	1010_2
$s =$	12	$=$	1100_2

leftmost 1 in binary rep. of s

Let $s = m_1 \oplus \cdots \oplus m_k$

Consider the column with the leftmost "1" in the binary rep. of s.

Note: At least one of the m_i also has a 1 in the same column.

Suppose that m_1 has a 1 in the same column.

Let's make a move in m_1: change m_1 to the number $(m_1 \oplus s)$.

Claims:

1. $m_1 \oplus s < m_1$
2. $(m_1 \oplus s)$ works, that is, if we set

   $m_1' = m_1 \oplus s$

   then $m_1' \oplus m_2 \oplus \cdots \oplus m_k = 0$.

Look at Claim 2:

New nim-sum is:

$$m_1' \oplus m_2 \oplus \cdots \oplus m_k = (m_1 \oplus s) \oplus m_2 \oplus \cdots \oplus m_k$$
$$= s \oplus (m_1 \oplus m_2 \oplus \cdots \oplus m_k)$$
$$= s \oplus s = 0.$$

Claim 1?

$m_1 \oplus s$ cancels off the 1 in the column where s has the leftmost 1.

$(m_1 \oplus s)$, in binary, looks the same as m_1 up to this column, and in this column, ~~all of~~ we change a 1 to a 0.

Therefore $(m_1 \oplus s) < m_1$!

E.g.

$$\begin{array}{rrcl}
 & 10 & = & 1010_2 \leftarrow m_1 \\
 & 13 & = & 1101_2 \\
 & 12 & = & 1100_2 \\
\oplus & 8 & = & 1000_2 \\
\hline
s \rightarrow & 3 & & 0011_2 \leftarrow s
\end{array}$$

leftmost 1 of s

$$m_1' = 10 \oplus 3$$
$$= \begin{array}{rr} & 1010_2 \\ \oplus & 11_2 \\ \hline & 1001_2 \end{array} = 9$$

Optimal move changes $10 \to 9$, forces the nim-sum to be 0.