MATH 2301 18/10/2023

* Last time : nim

Example $(3, 5, 10)$

$(3, 5, 10) \to (3, 1, 10)$

$(3, 5, 10) \to (5, 10)$ etc

(example moves)

* 2 piles of equal size = P position.

* Last time : Saw that you write each pile size in binary, take XOR.

$$\begin{array}{rcr} 3 & = & 11_2 \\ 5 & = & 101_2 \\ \oplus\ 10 & = & \oplus\ 1010_2 \\ \hline 12 & & 1100_2 \end{array}$$

nim-sum

XOR:
$1 \oplus 1 = 0 \oplus 0 = 0$
$1 \oplus 0 = 0 \oplus 1 = 1$
"exclusive OR"

Theorem : Let $(m_1, \ldots, m_k)$ be a nim position.
It is an N-position iff $m_1 \oplus \cdots \oplus m_k > 0$
It is a P-position iff $m_1 \oplus \cdots \oplus m_k = 0$.

Note : The nim-sum of $(m_1, \ldots, m_k)$ is zero iff each column in the binary representations has an even number of 1s.

E.g.

$$\begin{array}{rrcr} & 3 & = & 11_2 \\ & 5 & & 101_2 \\ \oplus & 10 & & \oplus\ 1010_2 \\ \hline & 12 & & 1100_2 \end{array}$$

vs

$$\begin{array}{rrcr} & 3 & = & 11_2 \\ & 5 & = & 101_2 \\ \oplus & 6 & = & 110_2 \\ \hline & 0 & & 000_2 \end{array}$$

Notice: To make nim-sum 0, you need to cancel ~~the l~~ (in particular) the leftmost 1 from the nim-sum

E.g.

$$\begin{array}{rrcl} & 10 & = & \boxed{1010}_2 \\ & 13 & = & 1101_2 \\ & 12 & = & 1100_2 \\ \oplus & 8 & = & 1000_2 \\ \hline & 3 & = & \boxed{0011}_2 \end{array}$$

← make a move

Pf sketch of theorem:

$(m_1, \dots, m_k)$ is some nim-position.

Case 1: Suppose that $m_1 \oplus \dots \oplus m_k = 0$.

Properties of the nim-sum

1. $x \oplus x = 0$
2. If $x \oplus y = z$ then
$$x \oplus y \oplus y = z \oplus y$$
i.e. $x = z \oplus y = y \oplus z$

E.g. $(1, 2, 3)$

$$\begin{array}{rrcr} & 1 & = & 01_2 \\ & 2 & = & 10_2 \\ \oplus & 3 & = & 11_2 \\ \hline & & & 00_2 \end{array}$$

If $(m_1, \dots, m_k)$ is such that $m_1 \oplus \dots \oplus m_k = 0$ then, any move results in a non-zero nim-sum.

Pf: If you make a move in the first pile (say) $m_1 \longmapsto m_1'$ (by eating some smarties)

New nim-sum is:
$$(m_1' \oplus m_2 \oplus m_3 \oplus \dots \oplus m_k) = s$$

$$s = s \oplus \underbrace{(m_1 \oplus m_2 \oplus \dots \oplus m_k)}_{= 0}$$

$$s = (m_1' \oplus \cancel{m_2} \oplus \cdots \oplus \cancel{m_k}) \oplus (m_1 \oplus \cancel{\ } \cdots \oplus \cancel{m_k})$$

$$s = m_1' \oplus m_1$$

Note: $x \oplus y = 0$ iff $x = y$.

So, since $m_1' \neq m_1$, we see $s \neq 0$.

$\Rightarrow$ Case 1 done: If $(m_1 \oplus \cdots \oplus m_k) = 0$, then any move makes the nim-sum > 0.

Case 2: $(m_1, \ldots, m_k)$ a position such that $m_1 \oplus \cdots \oplus m_k > 0$.

E.g.

$$\begin{array}{rrcr} & 3 & = & 11_2 \\ & 5 & = & 101_2 \\ m_1 = \oplus & 10 & = & 1010_2 \\ \hline s = & 12 & = & 1100_2 \end{array}$$

leftmost 1 in binary rep. of s

Let $s = m_1 \oplus \cdots \oplus m_k$

Consider the column with the leftmost "1" in the binary rep. of s.

Note: At least one of the m_i also has a 1 in the same column.

Suppose that m_1 has a 1 in the same column.

Let's make a move in m_1: change m_1 to the number $(m_1 \oplus s)$.

Claims:

(1) $m_1 \oplus s < m_1$

(2) $(m_1 \oplus s)$ works, that is, if we set

$$m_1' = m_1 \oplus s$$

then $m_1' \oplus m_2 \oplus \cdots \oplus m_k = 0$.

Look at Claim 2:

New nim-sum is:

$$m_1' \oplus m_2 \oplus \cdots \oplus m_k = (m_1 \oplus s) \oplus m_2 \oplus \cdots \oplus m_k$$

$$= s \oplus (m_1 \oplus m_2 \oplus \cdots \oplus m_k)$$

$$= s \oplus s = 0.$$

Claim 1?

$m_1 \oplus s$ cancels off the 1 in the column where s has the leftmost 1.

$(m_1 \oplus s)$, in binary, looks the same as m_1 up to this column, and in this column, we change a 1 to a 0.

Therefore $(m_1 \oplus s) < m_1$!

E.g.

$$\begin{array}{rcl} 10 & = & 1010_2 \leftarrow m_1 \\ 13 & = & 1101_2 \\ 12 & = & 1100_2 \\ \oplus \ 8 & = & 1000_2 \\ \hline s \rightarrow 3 & & 0011_2 \leftarrow s \end{array}$$

leftmost 1 of s

$$\begin{aligned} m_1' &= 10 \oplus 3 \\ &= \begin{array}{r} 1010_2 \\ \oplus \quad 11_2 \\ \hline 1001_2 \end{array} = 9 \end{aligned}$$

Optimal move changes $10 \to 9$, forces the nim-sum to be 0.