MATH 2301

23/10/2023

* EAP arrangements update : They will be handled centrally, but please send me a reminder email with your EAP so I can forward it to the exam office just to be doubly sure.

* Final office hrs : Posted on Wattle

* Last time : sums & equivalence of games

Def: We say $G_1 \sim G_2$ if for any H, the games $G_1 + H$ and $G_2 + H$ have the same outcome.

Game $\equiv$ impartial combinatorial game.

Rmk: If $G_1 \sim G_2$ then G_1 & G_2 have the same outcome. (To see this, add an empty game to both sides, i.e, take $H =$ empty game)

Rmk: The relation $\sim$ is an equivalence relation.

(Exercise !)

Warning: Even if G_1 and G_2 have the same outcome, they are not necessarily equivalent!

Example: G_1 is chomp $\boxed{\cdot\,|\,}$ $\rightsquigarrow$ N position.

G_2 is nim $(1, 2)$ $\rightsquigarrow$ N position

Claim: $G_1 \not\approx G_2$. How to show this?

Find a game H, such that $G_1 + H$ and $G_2 + H$ have different outcomes.

Hard question: how to find such an H???

Some experiments

Idea 1: Take $H = G_1$

We know: $G_1 + G_1$ is a P-position

(Two copies of the same game is always P, because P2 can mirror.)

What about $G_2 + G_1 \stackrel{?}{=} G_1 + G_2$?

$(\boxed{\cdot\,|\,}, (1,2)) \longrightarrow$ This ~~a~~ is an N-position:

because ...

$\downarrow$

$(\boxed{\cdot\,|\,}, (1)) \longrightarrow$ This is P by inspection.

Since $G_1 + H$ & $G_2 + H$ have different outcomes for $H = G_1$, and so $G_1 \not\approx G_2$.

Lemma: Let G be any game.
Let L be any game which is in a P-position.

Then, $G + L \sim G$.

Pf: We have to check the following (crazy) thing:
If H is any game, then we want $(G+L)+H$ and $G+H$ to have the same outcome.

Let us check this.

Let S be the game $G+H$.
Check: $S+L$ and S have the same outcome.

Case 1: Suppose S is an N-position
Winning strategy for $S+L$
- P1 makes the optimal move in S.
- P2 can either move in L, or in (what happens to) S; but both are P states.
- So, P1 can counter in each case to produce a (P, P) type state again.

$\Rightarrow$ $S+L$ is also an N-position!

Case 2 : Suppose S is a P-position.

Let's check that $S+L$ is also a P-position.

P2 has a winning strategy as follows:

- If P1 moves in S, P2 counters in S
- If P1 moves in L, P2 counters in L

Since both S & L are P-states, P2 wins.

$\Rightarrow$ Lemma, which says that $\boxed{G+L \sim G}$ for any losing (i.e P-position) game L.

Theorem : Let G_1 and G_2 be games. Then, $G_1 \sim G_2$ if and only if $G_1+G_2 \sim \phi$, which is true if and only if G_1+G_2 is a P-position.

~~Pf: Suppose $G_1 \sim G_2$. Using Lemma we know that~~

Pf: Suppose $G_1 \sim G_2$. We want to show that $G_1+G_2 \sim \phi$.

Let H be any game. Consider G_1+G_2+H vs $\phi+H = H$.

The game $G_1 + G_2 + H \cdot = G_1 + (G_2 + H)$

has the same outcome as $G_2 + (G_2 + H)$.

(b/c $G_1 \sim G_2$).

$$(G_2 + G_2 + H) = \underbrace{(G_2 + G_2)}_{\text{is a P-position!}} + H$$

Lemma $\Rightarrow$ $(G_2 + G_2) + H$ has the same outcome as H!

$\Rightarrow$ $G_1 + G_2 \sim \phi$, finishes one direction of iff

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Other direction, assume $G_1 + G_2 \sim \phi$

Need to show that $G_1 \sim G_2$ $\leadsto$ Next time