

* Admin: Final exam practice problems posted.
Solutions will NOT be posted
But, I have plenty of office hrs, so
come and check your answers!

* Last time: Criterion to check game equivalence \rightarrow pf today.

Theorem: $G_1 \sim G_2$ if and only if $G_1 + G_2 \sim \emptyset$
if and only if $G_1 + G_2$ is a P-state.

Pf sketch

① Suppose $G_1 \sim G_2$. Consider any H .

Then $G_1 + (G_2 + H)$ has the same outcome as:

$G_2 + (G_1 + H)$, which has the same outcome as H .

($G_2 + G_2$ is a P-state; adding P-states does not change the outcome \rightarrow see lemma from last time.) \parallel
($\emptyset + H$)

$\Rightarrow G_1 + G_2 + H \neq \emptyset + H = H$ have same outcome.

② Suppose $G_1 + G_2 \sim \emptyset$. Consider any H .

Want: $G_1 + H$ and $G_2 + H$ to have the same outcome.

$G_1 + H$ has the same outcome as $(G_1 + G_2) + (G_1 + H)$

$(G_1 + G_2) + (G_1 + H)$ is the same as $\underbrace{(G_1 + G_1)}_{\text{P-state}} + (G_2 + H)$

Has the same outcome as $G_2 + H$.

③ Finally if $G_1 + G_2 \sim \emptyset$ then $G_1 + G_2$ is a P-state
(Equivalent games have the same outcome!)

Now if $G_1 + G_2$ is a P-state, then $G_1 + G_2 \sim \emptyset$
(See Lemma from yesterday: Adding a P-state to a game doesn't change its outcome.)

Today: Fixing N & P labelling

The problem: It is not compatible with game sum.

What we know:

① If G is P-state & H is a P-state, then $G + H$ is a P-state.

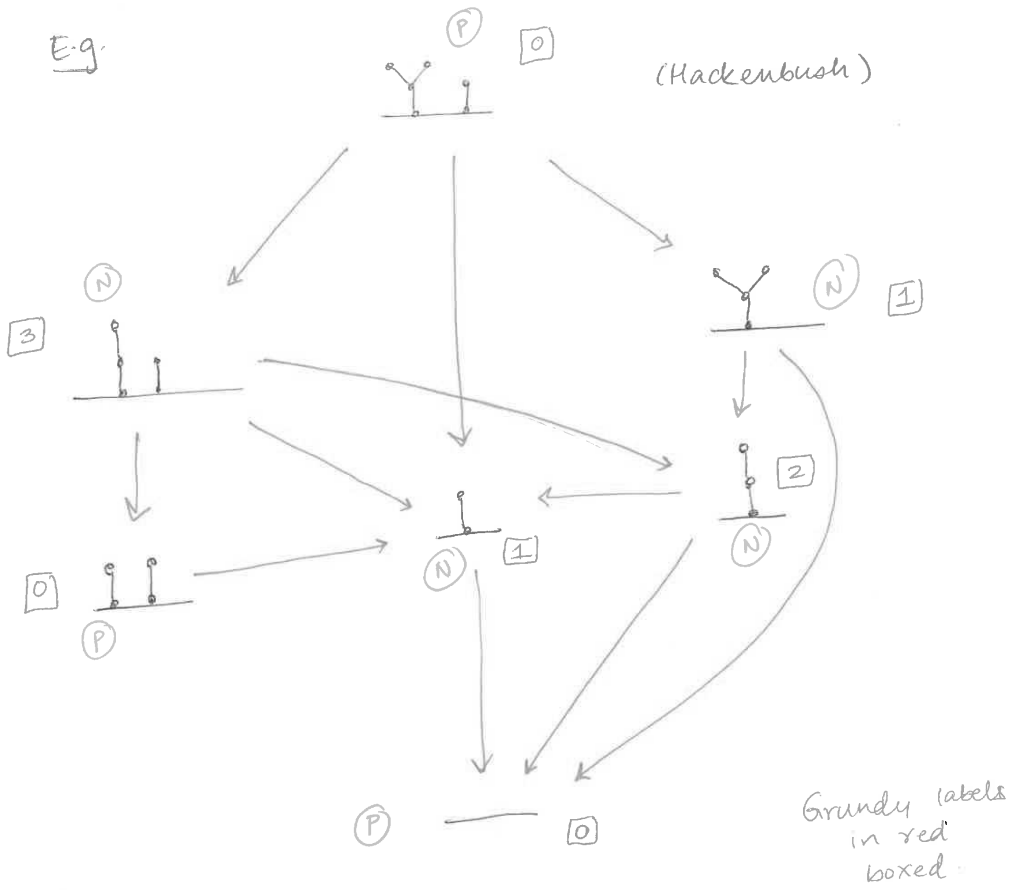
② If G is N and H is P (or vice-versa), then $G + H$ is N-state.

③ If G is an N-state & H is an N-state, then you don't know the outcome of $G + H$!

(compare $\boxed{1} + (1,2)$ vs $\boxed{1} + \boxed{1}$)
chomp + nim chomp + chomp

* Grundy labelling.

E.g.



- * Label each terminal position by 0.
- * Consider a position which points to n positions with Grundy labels $\{g_1, g_2, \dots, g_n\}$ (by arrows). Label this position by $\text{mex}\{g_1, g_2, \dots, g_n\}$
 = minimum excluded
 = minimum number in $\{0, 1, 2, \dots\}$ which does not appear as one of the g_i .

(3)

* Observe

All P-states have a Grundy value of 0.
 All N-states have a Grundy value > 0 .

~~Full~~ Full proof skipped.

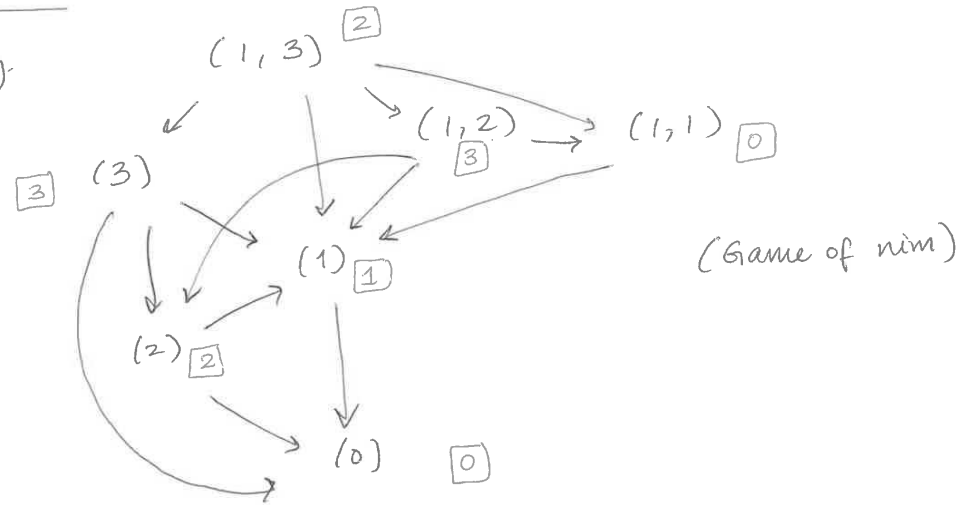
Idea: Start labelling a graph by N/P and Grundy simultaneously.

Terminal positions get P / 0

As you move up the graph, see that N-positions (those that point to at least one P) get positive labels.

~~But~~ On the other hand, P-positions are those that only point to N-positions, so their Grundy value is a mex of all positive numbers $\Rightarrow 0$.

E.g.



Observation

(Mysterious): The Grundy labels appear to be the same as the nim-sum of each position.

(4)