MATH 2301

25/10/2023

* Admin : Final exam practice problems posted.
Solutions will NOT be posted
But, I have plenty of office hrs, so
come and check your answers!

* Last time : Criterion to check game equivalence → pf today.

Theorem : $G_1 \sim G_2$ if and only if $G_1 + G_2 \sim \phi$
if and only if $G_1 + G_2$ is a P-state.

Pf sketch

① Suppose $G_1 \sim G_2$. Consider any H.
Then $G_1 + (G_2 + H)$ has the same outcome as :
$G_2 + (G_2 + H)$, which has the same outcome as H.
$\parallel$
($G_2 + G_2$ is a P-state ; adding P-states does not change the outcome → see lemma from last time.) $(\phi + H)$

$\Rightarrow$ $G_1 + G_2 + H$ & $\phi + H = H$ have same outcome.

② Suppose $G_1 + G_2 \sim \phi$. Consider any H.
Want : $G_1 + H$ and $G_2 + H$ to have the same outcome.
$G_1 + H$ has the same outcome as $(G_1 + G_2) + (G_1 + H)$
$(G_1 + G_2) + (G_1 + H)$ is the same as $\underbrace{(G_1 + G_1)}_{\text{P-state}} + (G_2 + H)$
Has the same outcome as $G_2 + H$.

③ Finally if $G_1 + G_2 \sim \phi$ then $G_1 + G_2$ is a P-state

(Equivalent games have the same outcome!)

Now if $G_1 + G_2$ is a P-state, then $G_1 + G_2 \sim \phi$

(See Lemma from yesterday: Adding a P-state to a game doesn't change its outcome.)

<u>Today</u>: Fixing N & P labelling

<u>The problem</u>: It is not compatible with game sum.

What we know:

① If G is P-state & H is a P-state, then G+H is a P-state.

② If G is N and H is P (or vice-versa), then G+H is N-state.

③ If G is an N-state & H is an N-state, then you don't know the outcome of G & H!

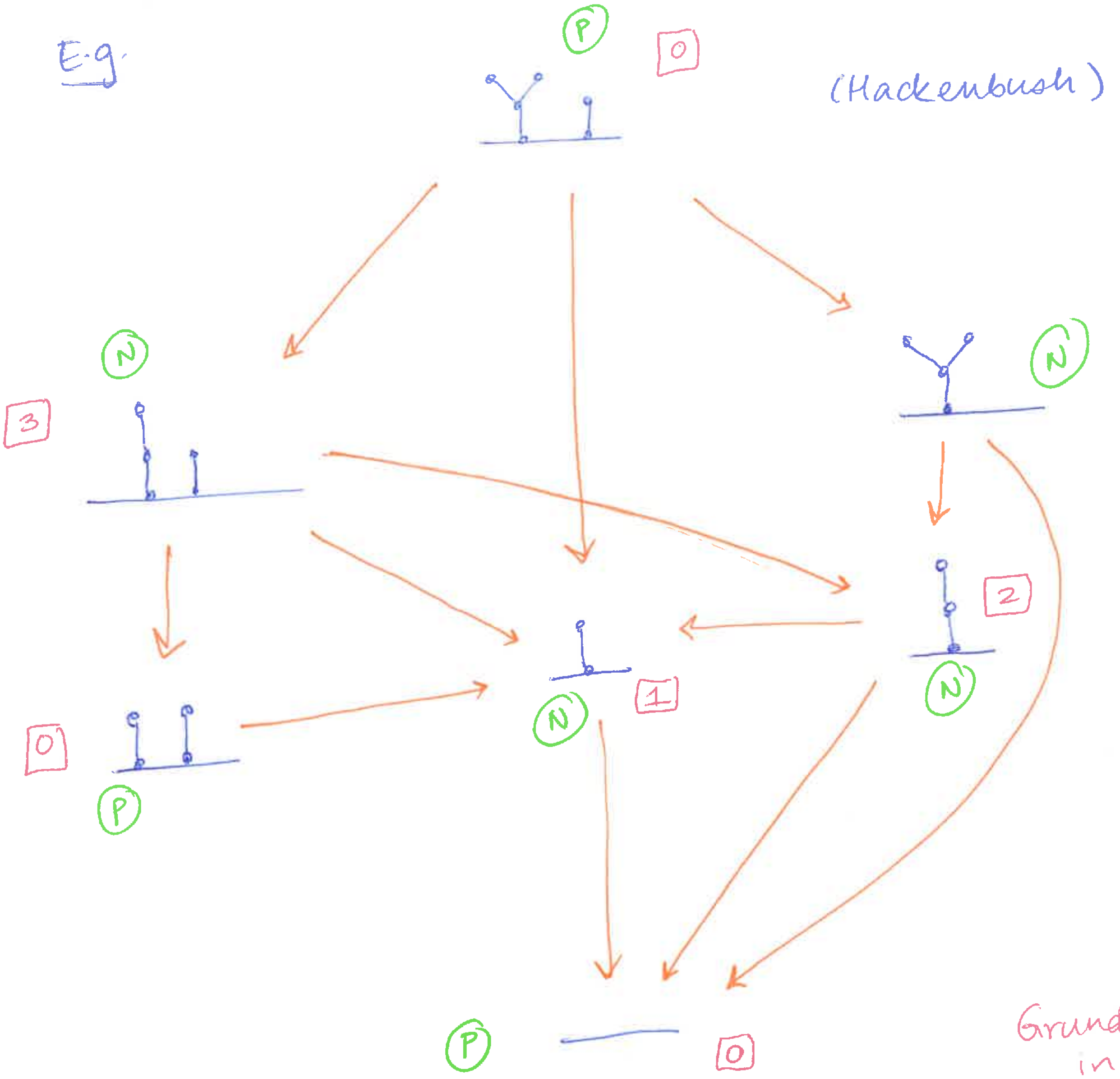
(compare  + (1,2) vs  + )

chomp + nim chomp + chomp

* Grundy labelling.

E.g.

(Hackenbush)



Grundy labels in red boxed.

* Label each terminal position by 0.
* Consider a position which points to (by arrows) positions with Grundy labels $\{g_1, g_2, \ldots, g_k\}$

  Label this position by $\text{mex}\,\{g_1, g_2, \ldots, g_k\}$

  mex = minimum excluded

  = minimum number in $\{0, 1, 2, \ldots\}$ which does not appear as one of the g_i.

* **Observe**

All P- states have a Grundy value of 0.

All N- states have a Grundy value > 0.

Full proof skipped.

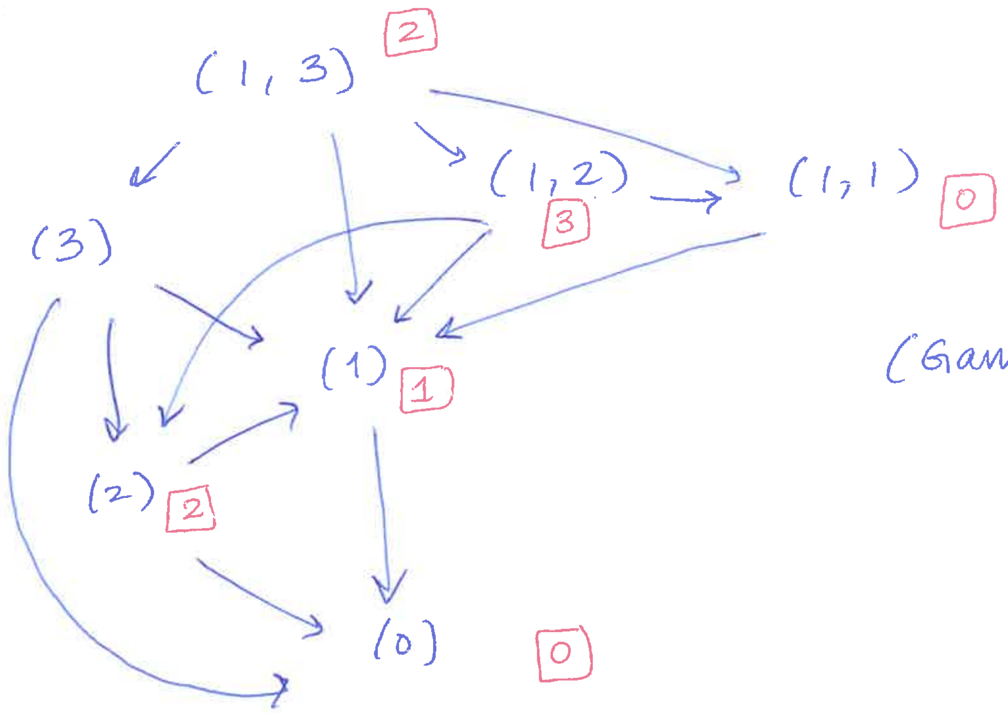
Idea: Start labelling a graph by N/P and Grundy simultaneously.

Terminal positions get P / 0

As you move up the graph, see that N-positions (those that point to at least one P) get positive labels.

On the other hand, P-positions are those that only point to N-positions, so their Grundy value is a mex of all positive numbers $\Rightarrow$ 0.

E.g.



(Game of nim)

Observation (Mysterious): The Grundy labels appear to be the same as the nim-sum of each position.