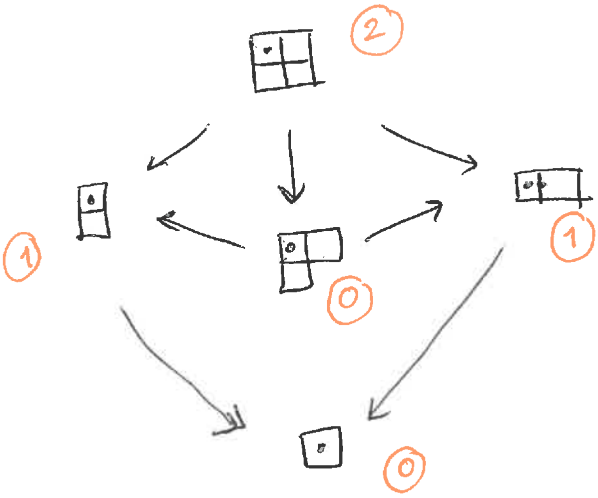
MATH2301 27/10/2023

* Happy end of term! See you @ office hrs... Check Wattle for times. We'll either be in HN 4.84 or in one of the meeting rooms near the common area/kitchen of HN 4th floor.

* Last time: Grundy labelling

E.g. (Chomp)



E.g. Single-pile nim game with m smarties in one pile has Grundy label m.

if $m = 4$: $\overset{\boxed{4}}{4} \to \overset{\boxed{3}}{3} \to \overset{\boxed{2}}{2} \to \overset{\boxed{1}}{1} \to \overset{\boxed{0}}{0}$

Notation: We call the ~~g~~ single-pile nim game with m objects as $*m$.

So $\mathrm{Grundy}(*m) = m$.

* **Recall**: Grundy $(G) = 0$ iff G is a P-position

Grundy $(G) > 0$ iff G is an N-position.

But wait, there's more!

* **Theorem**: Let G & H be two games. Then...

the Grundy value of $G+H$ equals the nim-sum of the Grundy values of G & H

$$\text{Grundy}(G+H) = \text{Grundy}(G) \oplus \text{Grundy}(H) \ !$$

Special case: If $G = (m_1, \ldots, m_k)$ is a nim game, then

$$G = (*m_1) + (*m_2) + \cdots + (*m_k)$$

game sum

Thm $\Rightarrow$ $\text{Grundy}(G) = (m_1 \oplus m_2 \oplus \cdots \oplus m_k)$

(Explains labelling from Wed.)

To prove this: we need to show:

$$\begin{tikzcd}
 & \overset{g \quad h}{(G+H)} \arrow[dl] \arrow[d] \arrow[dr] \arrow[r] & \cdots \\
\underset{g' \oplus h}{(G'+H)} & \underset{g'' \oplus h'}{(G''+H)} & \underset{g \oplus h'}{(G+H')}
\end{tikzcd}$$

(schematic of portion of game graph)

We want: $\text{mex}\{g' \oplus h, g \oplus h'\} = g \oplus h$

Pf sketch

Let $g = \text{Grundy}(G)$ & $h = \text{Grundy}(H)$.

Let $s = g \oplus h$.

Want $s = \text{mex}\{g' \oplus h,\ g \oplus h' \mid g' = \text{Grundy}(G')$ for $G \to G'$ valid move; $h' = \text{Grundy}(H)$ for $H \to H'$ valid move$\}$

(By induction, we may assume result for $G'+H$, $G+H'$)

That is, we need to show that:

① If we make any move in $G+H$, we don't get the Grundy value/nim sum to be s,

② If $s' < s$ then we can make a move in $G+H$ to get to s'.

(Pf by example): Suppose $\text{Grundy}(G) = 5$, $\text{Grundy}(H) = 11$

$s = 5 \oplus 11$:

$$\begin{array}{r} 101_2 \\ \oplus\ 1011_2 \\ \hline 1110_2 \end{array}$$

$s = 14 = 1110_2$

① Note: If we move in G, we get a Grundy value $\neq 5$
" " " " H, " " " " " $\neq 11$

$\Rightarrow$ new nim-sum after we make a move will not equal 14

②: Choose any $s' < 14$, e.g. $s' = 4$.

We now need to find a move in G OR H, such that the new nim-sum of Grundy values equals 4.

Grundy $(G) = 5$ Grundy $(H) = 11$

$$\begin{array}{rcll} 101_2 & = & 5 & g \\ \oplus 1011_2 & = & 11 & h \\ \hline 1110_2 & = & 14 & s \end{array}$$

$s' = 4 = 100_2$

Need to change $g \to g'$ or $h \to h'$ such that $g' \oplus h$ (resp. $g \oplus h'$) equals 4.

Recall: G points to states labelled $0, 1, 2, 3, 4$ (maybe others, but not 5)

H points to states labelled $0, 1, \dots, 10$ (maybe others, but not 11).

Note: Sending H to H' labelled 1 will make the new nim-sum equal to 4:

$$\begin{array}{r} 101_2 \\ 1_2 \\ \hline 100_2 \end{array} = 4.$$

Another example: Take $s' = 9 = 1001_2$

$$\begin{array}{rl} 101_2 & \\ \oplus 1011_2 & \\ \hline 1110_2 & s \end{array}$$

1001_2 s'

$$\begin{array}{r} 10_2 \\ 1011_2 \\ \hline 1001_2 \end{array}$$

Explanation → (hard) exercise

Idea: Look at the first column on left where s & s' differ. Since $s' < s$, s will have 1 and s' a 0. Use idea from previous nim pf to figure out how to cancel that 1.

Theorem (Sprague–Grundy theorem):

Let G be any impartial combinatorial game.

Let g be $\text{Grundy}(G)$.

Then $G \sim (*g)$.

(Motto: Every game is some form of nim!)

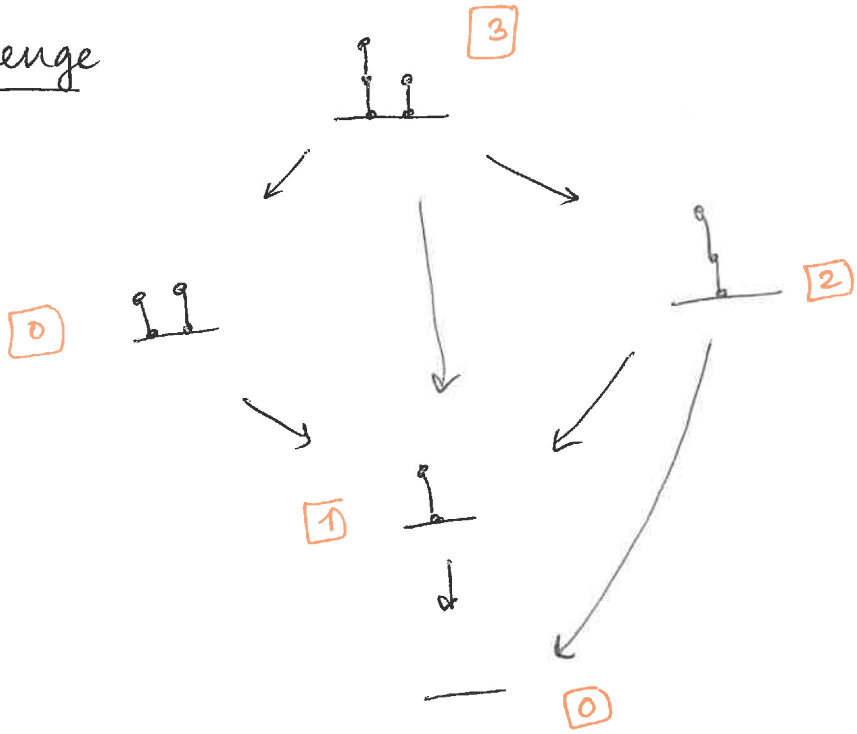
Pf: $G \sim (*g)$ iff $G + (*g)$ is a P-position.

$$\text{Grundy}(G + (*g)) = \text{Grundy}(G) \oplus \text{Grundy}(*g)$$
$$= g \oplus g = 0.$$

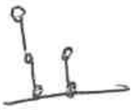
$\Leftrightarrow$ $G + (*g)$ is a P-position. Done!

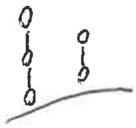
$\square$

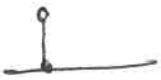
Eg / Challenge



S-G thm $\Rightarrow$  $\sim *3$

$\Rightarrow$  $+ (*3)$ is a P-position.

 $+ (*1)$

 $+ (*1)$

— $+ (*1)$

— $+ (*0)$