Andrei Negut

Moduli spaces of sheaves: geometry and representation theory (1/3)

9/7/18
SSGRT

idea: $\mathcal{A}$ "$\curvearrowright$" K_r

goals: - describe $\mathcal{A}$ = elliptic Hall algebra
- describe K_r = algebraic K-theory of moduli of stable sheaves on S

smooth projective

- define "action"
- prove main idea

context: framework of Nakajima, Grojnowski, Ginzburg, Varagnolo - Vasserot ...

but specifically the study of Hall algebras

even specificallier, the K-theoretic Hall algebra of Schiffmann - Vasserot, Minets, Sala ...

- $S = T^*C$, this is connected w/ Higgs bundles

plan: - lectures 2&3: go through goals
- lecture 1: ~~defining $\mathcal{M} = \mathcal{M}_r$ = space of rank r sheaves on S~~

(read lecture 1 of my CIME lecture notes)

- lecture 1: give an application of main idea = = rep. theory of Hilb(K3)

$S/_{\mathbb{C}}$ a K3 surface, features

- smooth projective
- $K_S \cong \mathcal{O}_S$

$\mathbb{Q}$ coeff.

$$H^{2*}(S) = \underbrace{H^0(S)}_{\text{1-dim}} \oplus \underbrace{H^2_{alg}(S)}_{\substack{\text{22-6 dim}\\ c_1(\text{line bundles})}} \oplus \underbrace{H^2_{tr}(S)}_{\substack{\text{6 dim}}} \oplus \underbrace{H^4(S)}_{\text{1-dim}}$$

6 dim orthogonal complement of $H^2_{alg}(S)$ w.r.t.

$\langle \cdot , \cdot \rangle : H^*(S) \otimes H^{4-*}(S) \to \mathbb{Q}$

Chow groups

grading by codimension

$$\begin{tikzcd} A^*(S) = \left\{ \begin{array}{c} \mathbb{Q}\text{-linear sums} \\ \text{of algebraic cycles} \end{array} \right\} \Big/ \text{rat. equivalence} \arrow[d, "\xi"', "\text{cycle map}"] \\ H^{2*}(S) \end{tikzcd}$$

Def (Beauville-Voisin) $R(S) := A^*_{div}(S) \subset A^*(S)$

subring generated by divisor classes $= c_1(\text{line bundle})$

$\exists$ a given cycle $c \in A^2(S)$ s.t.

$$\underset{A^1(S)}{\ell} \cdot \underset{A^1(S)}{\ell'} = \langle \xi(\ell), \xi(\ell') \rangle c \qquad \forall \ell, \ell' \in A^1(S)$$

$$\dim_{\mathbb{Q}} R(S) = \underset{A^0(S)}{1} + \overbrace{22-6} + \underset{A^2(S)}{\overset{\text{comes from } \mathbb{Q}\cdot c}{1}}$$

$\dim_{\mathbb{Q}} A^2(S) =$ very ∞

Thm (B-V): $\xi : \underset{A^*_{div}(S)}{R(S)} \hookrightarrow H^{2*}(S)$ is injective.

$$\mathrm{Hilb} = \left\{ \begin{array}{l} \text{ideals, finite colength} \\ I \subset \mathcal{O}_S \end{array} \right\}$$

$$\cup$$

$$\mathrm{Hilb}_n = \left\{ \text{—}\|\text{——} \quad \text{when } \operatorname{codim}_{\mathbb{C}} \frac{\mathcal{O}_S}{I} = n \right\}$$

$\mathrm{Hilb}_0 = pt$

$\mathrm{Hilb}_1 = S$

$\mathrm{Hilb}_2 = Bl_\Delta (S \times S) / \mathfrak{S}_2$

$$A^*(\mathrm{Hilb}) = \bigoplus_{n=0}^{\infty} A^*(\mathrm{Hilb}_n)$$

with $\mathbb{Q}$-coefficients

Def: $A^*_{div}(\mathrm{Hilb}) \subset A^*(\mathrm{Hilb})$ subring generated by $A^1(\mathrm{Hilb})$

Conjecture: (B-V): $$\begin{tikzcd} \varphi : A^*_{div}(\mathrm{Hilb}) \arrow[d, hook] \\ H^{2*}(\mathrm{Hilb}) \end{tikzcd}$$

(expected for any irreducible holomorphic symplectic variety → these correspond to simply connected HK varieties)

Goal: joint w/ Maulik, give a rep^n theory proof of conjecture.

Def:
$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, phantom, "\parallel"] & \mathrm{Hilb} \times S \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \left\{ (I, x) \text{ s.t. } x \in \mathrm{supp}\left(\frac{\mathcal{O}_S}{I}\right) \right\} & \mathrm{Hilb} & S \end{tikzcd}$$

universal subscheme (codimension 2)

a tautological class $= \pi_{1*}\left(ch_k(\mathcal{O}_Z) \cdot \pi_2^* \gamma\right)$ $\forall k \in \mathbb{N}$ $\forall \gamma \in RIS$

$$\begin{tikzcd} A^*_{taut}(\mathrm{Hilb}) \arrow[r, phantom, "\subset"] & A^*(\mathrm{Hilb}) \\ A^*_{div}(\mathrm{Hilb}) \arrow[u, phantom, "\cup"] & \end{tikzcd}$$

generated by tautological classes

<u>**Thm**</u> (Maulik - N.)

$\xi : A^*_{taut}(\mathrm{Hilb}) \longrightarrow H^{2*}(\mathrm{Hilb})$ is injective

Prove using repⁿ theory!

Def:

$$\mathrm{Hilb}_{d,d+n} = \left\{ (I' \subset I) \;\middle|\; \mathrm{supp}\, \frac{I}{I'} \text{ in a single point } x \in S \right\} \subset \underset{\overset{\cap}{I'}}{\mathrm{Hilb}_{d+n}} \times \underset{\overset{\cap}{I}}{\mathrm{Hilb}_d}$$

$$\begin{tikzcd} & \mathrm{Hilb}_{d,d+n} \arrow[dl, "p_-"'] \arrow[d, "p_S"] \arrow[dr, "p_+"] & \\ \mathrm{Hilb}_d & S & \mathrm{Hilb}_{d+n} \end{tikzcd}$$

(Nakajima, Grojnowski)

$a_{\pm n} : A^*(\mathrm{Hilb}) \longrightarrow A^*(\mathrm{Hilb} \times S)$

$n \in \mathbb{N}$

satisfy Heisenberg algebra rel:

$$a_n = (p_+ \times p_S)_* \circ p_-^*$$

$$a_{-n} = (-1)^{n-1} (p_- \times p_S)_* \circ p_+^*$$

$$\boxed{[a_n, a_{n'}] = n\, \delta^0_{n+n'} \cdot \Delta_* \circ \Pi_1^*}$$ as $A^*(\mathrm{Hilb}) \to A^*(\mathrm{Hilb} \times S \times S)$

a_n

$a_{n'}$

where $\Delta \cong S \hookrightarrow S \times S$

Ex

$\forall \gamma \in R(S)$

define $a_n(\gamma) : A^*(\mathrm{Hilb}) \longrightarrow A^*(\mathrm{Hilb})$

$$a_n(\gamma) = \Pi_{1*}\left(\Pi_2^* \gamma \cdot a_n\right)$$

$\Downarrow$

$$\boxed{[a_n(\gamma), a_{n'}(\gamma')] = n\, \delta^0_{n+n'} \langle \gamma, \gamma' \rangle \mathrm{Id}_{A^*(\mathrm{Hilb})}}$$

relations on $\mathrm{Heis}_{R(S)}$

inspired by Lehn,

$$A^*(\mathrm{Hilb}) \xrightarrow{L_n} A^*(\mathrm{Hilb})$$

$$L_n = \frac{1}{2} \sum_{\substack{k,l \in \mathbb{Z} \\ k+l=n}}' :a_k a_l: \underbrace{\left(\Delta - \sum e_i \boxtimes e^i\right)}_{\in A^*(S \times S)}$$

$\{e_i, e^i\}$ dual bases of $R(S)$

$$:a_k a_l: = \begin{cases} a_k a_l & \text{if } k \geq l \\ a_l a_k & \text{if } k < l \end{cases}$$

$A^*(S \times S) \ni \Gamma$

$a_k a_l(\Gamma)$ is defined as the composition below

$$\begin{tikzcd}
 & & & A^*(\mathrm{Hilb}) \\
A^*(\mathrm{Hilb}) \arrow[r, "a_l"] & A^*(\mathrm{Hilb} \times S) \arrow[r, "a_k \boxtimes \mathrm{Id}_S"] & A^*(\mathrm{Hilb} \times S \times S) \arrow[r, "\cdot \Pi_{23}^*(\Gamma)"] & A^*(\mathrm{Hilb} \times S \times S) \arrow[u, "\Pi_{1*}"']
\end{tikzcd}$$

__Prop:__ $[L_n, a_{n'}(\gamma')] = 0 \qquad \forall n, n'$
$\qquad\qquad\qquad\qquad\qquad \forall \gamma' \in R(S)$

$$[L_n, L_{n'}] = (n-n') \cdot L_{n+n'} + \delta^0_{n-n'} \frac{n^3-n}{12} \cdot b$$

rk of $H^2_{tr}(S)$

$\rightsquigarrow$ the operators $\{L_n, a_{n'}(\gamma')\}_{\substack{n,n' \in \mathbb{Z} \\ \gamma' \in R(S)}}$

give you an action of

$$\text{Vir} \times \text{Heis}_{R(S)} \curvearrowright A^*(\text{Hilb})$$

__Def:__ define $V \subset A^*(\text{Hilb})$ be the $\text{Vir} \times \text{Heis}_{R(S)}$-submodu generated by the vector $1 \in A^*(\underset{\overset{\shortparallel}{\text{pt}}}{\text{Hilb}_0}) \subset A^*(\text{Hilb})$

__Thm:__ V is preserved by multiplication with __tautological classes__ $V \supset A^*_{\text{taut}}(\text{Hilb})$.

$\hookrightarrow$ by this theorem, the main Thm reduces to the fact that $\xi: V \to H^{2*}(\text{Hilb})$ is injective.

Main theorem follows from the easily proved fact that ξ is $\text{Vir} \times \text{Heis}_{R(S)}$ and the fact that V is a __simple module__

rep^n theory of Heis is easy
rep^n theory of Vir done by Feigin - Fuchs