Moduli spaces of sheaves on surfaces - Hecke correspondences and representation theory

Andrei Neguț

In modern terms, enumerative geometry is the study of **moduli spaces**: instead of counting various geometric objects, one describes the set of such objects, which if lucky enough to enjoy good geometric properties is called a moduli space. For example, the moduli space of linear subspaces of $\mathbb{A}^n$ is the Grassmannian variety, which is a classical object in representation theory. Its cohomology and intersection theory (as well as those of its more complicated cousins, the flag varieties) have long been studied in connection with the Lie algebras $\mathfrak{sl}_n$.

The main point of this mini-course is to make the analogous connection between the moduli space $\mathcal{M}$ of certain more complicated objects, specifically sheaves on a smooth projective surface, with an algebraic structure called the elliptic Hall algebra $\mathcal{E}$ (see [2], [24]). We will recall the definitions of these objects in Sections 1 and 2, respectively, but we note that the algebra $\mathcal{E}$ is isomorphic to the quantum toroidal algebra, which is a central-extension and deformation of the Lie algebra $\mathfrak{gl}_1[s^{\pm 1}, t^{\pm 1}]$. Our main result is the following (see [20]):

Theorem 1. *There exists an action $\mathcal{E} \curvearrowright K_{\mathcal{M}}$, defined as in Subsection 2.6.*

One of the main reasons why one would expect the action $\mathcal{E} \curvearrowright K_{\mathcal{M}}$ is that it generalizes the famous Heisenberg algebra action ([9], [14]) on the cohomology of Hilbert schemes of points (see Subsection 2.1 for a review). In general, such actions are useful beyond the beauty of the structure involved: putting an algebra action on $K_{\mathcal{M}}$ allows one to use representation theory in order to describe various intersection-theoretic computations on $\mathcal{M}$, such as Euler characteristics of sheaves. This has far-reaching connections with mathematical physics, where numerous computations in gauge theory and string theory have recently been expressed in terms of the cohomology and K–theory groups of various moduli spaces (the particular case of the moduli space of stable sheaves on a surface leads to the well-known Donaldson invariants). Finally, we will give some hints as to how one would categorify the action of Theorem 1, by replacing the K–theory groups of $\mathcal{M}$ with derived categories of coherent sheaves. As shown in [7], [21], this categorification is closely connected to the Khovanov homology of knots in the 3-sphere or in solid tori, leading one to geometric knot invariants.

I would like to thank the organizers of the CIME School on Geometric Representation Theory and Gauge Theory: Ugo Bruzzo, Antonella Grassi and Francesco Sala, for making this wonderful event possible. Special thanks are due to Davesh Maulik and Francesco Sala for all their support along the way.

1 Moduli spaces of sheaves on surfaces

The contents of this Section require knowledge of algebraic varieties, sheaves and cohomology, and derived direct and inverse images of morphisms at the level of [10]. Let X be a projective variety over an algebraically closed field of characteristic zero, henceforth denoted by $\mathbb{C}$. We fix an embedding $X \hookrightarrow \mathbb{P}^N$, meaning that the tautological line bundle $\mathcal{O}(1)$ on projective space restricts to a very ample line bundle on X, which we denote by $\mathcal{O}_X(1)$. The purpose of this Section is to describe a scheme $\mathcal{M}$ which represents the functor of flat families of coherent sheaves on X, by which we mean the following things:

- for any scheme T, there is an identification:

$$\text{Maps}(T, \mathcal{M}) \cong \Big\{\mathcal{F} \text{ coherent sheaf on } T \times X \text{ which is flat over } T\Big\} \quad (1)$$

  which is functorial with respect to morphisms of schemes $T \to T'$

- there exists a **universal sheaf** $\mathcal{U}$ on $\mathcal{M} \times X$, by which we mean that the identification in the previous bullet is explicitly given by:

$$T \xrightarrow{\phi} \mathcal{M} \quad \leftrightsquigarrow \quad \mathcal{F} = (\phi \times \text{Id}_X)^*(\mathcal{U}) \quad (2)$$

A coherent sheaf $\mathcal{F}$ on $T \times X$ can be thought of as the family of its fibers over closed points $t \in T$, denoted by $\mathcal{F}_t := \mathcal{F}|_{t \times X}$. There are many reasons why one restricts attention to flat families, such as the fact that flatness implies that the numerical invariants of the coherent sheaves $\mathcal{F}_t$ are locally constant in t. We will now introduce the most important such invariant, the Hilbert polynomial.

1.1 Subschemes and Hilbert polynomials

A subscheme of X is the same thing as an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$, and many classical problems in algebraic geometry involve constructing moduli spaces of subschemes of X with certain properties.

Example 1. *If $X = \mathbb{P}^N$, the moduli space parametrizing k dimensional linear subspaces of $\mathbb{P}^N$ is the Grassmannian* $\text{Gr}(k+1, N+1)$.

We will often be interested in classifying subschemes of X with certain properties (in the example above, the relevant properties are dimension and linearity). Many of these properties can be read off algebraically from the ideal sheaf $\mathcal{I}$.

Definition 1. *The Hilbert polynomial of a coherent sheaf $\mathcal{F}$ on X is defined as:*

$$P_{\mathcal{F}}(n) = \dim_{\mathbb{C}} H^0(X, \mathcal{F}(n)) \quad (3)$$

for n large enough. We write $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{O}_X(n)$.

In the setting of the Definition above, the Serre vanishing theorem ensures that $H^i(X, \mathcal{F}(n)) = 0$ for $i \geq 1$ and n large enough, which implies that (3) is a polynomial in n. A simple exercise shows that if:

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is a short exact sequence of coherent sheaves on X, then:

$$P_{\mathcal{F}}(n) = P_{\mathcal{G}}(n) - P_{\mathcal{H}}(n)$$

Therefore, fixing the Hilbert polynomial of an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ is the same thing as fixing the Hilbert polynomial of the quotient $\mathcal{O}_X/\mathcal{I}$, if X is given.

Example 2. *If $X = \mathbb{P}^N$ and $\mathcal{I}$ is the ideal sheaf of a k-dimensional linear subspace, then $\mathcal{O}_X/\mathcal{I} \cong \mathcal{O}_{\mathbb{P}^k}$, which implies that:*

$$P_{\mathcal{O}_X/\mathcal{I}}(n) = \dim_{\mathbb{C}} H^0(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(n)) = \\ = \dim_{\mathbb{C}} \Big\{ \textit{degree } n \textit{ part of } \mathbb{C}[x_0, ..., x_k] \Big\} = \binom{n+k}{k}$$

If $\mathcal{I}$ is the ideal sheaf of an arbitrary subvariety of $\mathbb{P}^N$, the degree of the Hilbert polynomial $P_{\mathcal{O}_X/\mathcal{I}}$ is the dimension of the subscheme cut out by $\mathcal{I}$, while the leading order coefficient of $P_{\mathcal{O}_X/\mathcal{I}}$ encodes the degree of the said subscheme. Therefore, the Hilbert polynomial knows about geometric properties of subschemes.

1.2 Hilbert and Quot schemes

We have already seen that giving a subscheme of a projective variety X is the same thing as giving a surjective map $\mathcal{O}_X \twoheadrightarrow \mathcal{O}_X/\mathcal{I}$, and that such subschemes are parametrized by their Hilbert polynomials.

Definition 2. *There exists a moduli space parametrizing subschemes $\mathcal{I} \subset \mathcal{O}_X$ with fixed Hilbert polynomial $P(n)$, and it is called the **Hilbert scheme**:*

$$\mathrm{Hilb}_P = \Big\{ \mathcal{I} \subset \mathcal{O}_X \textit{ such that } P_{\mathcal{O}_X/\mathcal{I}}(n) = P(n) \textit{ for } n \gg 0 \Big\}$$

We also write:

$$\mathrm{Hilb} = \bigsqcup_{P \textit{ polynomial}} \mathrm{Hilb}_P$$

A particularly important case in the setting of our lecture notes is when the Hilbert polynomial $P(n)$ is constant, in which case the subschemes $\mathcal{O}_X/\mathcal{I}$ are finite length sheaves. More specifically, if $P(n) = d$ for some $d \in \mathbb{N}$, then Hilb_P parametrizes subschemes of d points on X. It is elementary to see that Definition 2 is the $\mathcal{V} = \mathcal{O}_X$ case of the following more general construction:

Definition 3. *Fix a coherent sheaf $\mathcal{V}$ on X and a polynomial $P(n)$. There exists a moduli space, called the **Quot scheme**, parametrizing quotients:*

$$\mathrm{Quot}_{\mathcal{V},P} = \Big\{ \mathcal{V} \twoheadrightarrow \mathcal{F} \textit{ such that } P_{\mathcal{F}}(n) = P(n) \textit{ for } n \gg 0 \Big\}$$

We also write:

$$\mathrm{Quot}_{\mathcal{V}} = \bigsqcup_{P \text{ polynomial}} \mathrm{Quot}_{\mathcal{V},P}$$

Definitions 2 and 3 concern the existence of projective varieties (denoted by Hilb and $\mathrm{Quot}_{\mathcal{V}}$, respectively) which represent the functors of flat families of ideal sheaves $\mathcal{I} \subset \mathcal{O}_X$ and quotients $\mathcal{V} \twoheadrightarrow \mathcal{F}$, respectively. In the language at the beginning of this Section, we have natural identifications:

$$\mathrm{Maps}(T, \mathrm{Hilb}) \cong \Big\{ \mathcal{I} \subset \mathcal{O}_{T \times X}, \text{ such that } \mathcal{I} \text{ is flat over } T \Big\} \tag{4}$$

$$\mathrm{Maps}(T, \mathrm{Quot}_{\mathcal{V}}) \cong \Big\{ \pi^*(\mathcal{V}) \twoheadrightarrow \mathcal{F}, \text{ such that } \mathcal{F} \text{ is flat over } T \Big\} \tag{5}$$

where $\mathcal{I}$ and $\mathcal{F}$ are coherent sheaves on $T \times X$, and $\pi : T \times X \to X$ is the standard projection. The flatness hypothesis on these coherent sheaves implies that the Hilbert polynomial of the fibers $\mathcal{O}_X/\mathcal{I}_t$ and $\mathcal{F}_t$ are locally constant functions of the closed point $t \in T$. If these Hilbert polynomials are equal to a given polynomial P, then the corresponding maps in (4) and (5) land in the connected components $\mathrm{Hilb}_P \subset \mathrm{Hilb}$ and $\mathrm{Quot}_{\mathcal{V},P} \subset \mathrm{Quot}_{\mathcal{V}}$, respectively.

The construction of the schemes Hilb_P and $\mathrm{Quot}_{\mathcal{V},P}$ is explained in Chapter 2 of [11], where the authors also show that (2) is satisfied. Explicitly, there exist universal sheaves $\mathcal{I}$ on $\mathrm{Hilb} \times X$ and $\mathcal{F}$ on $\mathrm{Quot}_{\mathcal{V}} \times X$ such that the identifications (4) and (5) are given by sending a map $\phi : T \to \mathrm{Hilb}, \mathrm{Quot}_{\mathcal{V}}$ to the pull-back of the universal sheaves under ϕ.

Example 3. *Let us take $X = \mathbb{P}^1$ and consider zero-dimensional subschemes of X. Any such subscheme Z has finite length as an $\mathcal{O}_X$-module, so we may assume this length to some $d \in \mathbb{N}$. The ideal sheaf of Z is locally principal, hence there exist $[a_1 : b_1], ..., [a_d : b_d] \in \mathbb{P}^1$ such that $\mathcal{I}$ is generated by:*

$$(sa_1 - tb_1)...(sa_d - tb_d)$$

where $\mathbb{C}[s,t]$ is the homogeneous coordinate ring of X. Therefore, length d subschemes of $\mathbb{P}^1$ are in one-to-one correspondence with degree d homogeneous polynomials in s, t (up to scalar multiple) and so it should not be a surprise that:

$$\mathrm{Hilb}_d \cong \mathbb{P}^d \tag{6}$$

A similar picture holds when X is a smooth curve C, and the isomorphism (6) *holds locally on the Hilbert scheme of length d subschemes of C.*

1.3 Moduli space of sheaves

If X is a projective variety, the only automorphisms of $\mathcal{O}_X$ are scalars (elements of the ground field $\mathbb{C}$). Because of this, $\mathrm{Hilb} = \mathrm{Quot}_{\mathcal{O}_X}$ is the moduli space of coherent sheaves of the form $\mathcal{F} = \mathcal{O}_X/\mathcal{I}$. Not all coherent sheaves are of this form, e.g. $\mathcal{F} = \mathbb{C}_x \oplus \mathbb{C}_x$ cannot be written as a quotient of $\mathcal{O}_X$ for any closed point $x \in X$. However, Serre's theorem implies that all coherent sheaves $\mathcal{F}$ with fixed Hilbert polynomial can be written as quotients:

$$\phi : \mathcal{O}_X(-n)^{P(n)} \twoheadrightarrow \mathcal{F} \tag{7}$$

for some large enough n, where $\mathcal{O}_X(1)$ is the very ample line bundle on X induced from the embedding of $X \hookrightarrow \mathbb{P}^N$ (the existence of (7) stems from the fact that $\mathcal{F}(n)$ is generated by global sections, and its vector space of sections has dimension $P(n)$). Therefore, intuitively one expects that the "scheme":

$$\mathcal{M}_P \text{``} := \text{''} \Big\{ \mathcal{F} \text{ coherent sheaf on } X \text{ with Hilbert polynomial } P \Big\} \tag{8}$$

(in more detail, $\mathcal{M}_P$ should be a scheme with the property that $\text{Maps}(T, \mathcal{M}_P)$ is naturally identified with the set of coherent sheaves $\mathcal{F}$ on $T \times X$ which are flat over t, and the Hilbert polynomial of the fibers $\mathcal{F}_t$ is given by P) satisfies:

$$\mathcal{M}_P = \text{Quot}_{\mathcal{O}_X(-n)^{P(n)},P} / GL_{P(n)} \tag{9}$$

where $g \in GL_{P(n)}$ acts on a homomorphism ϕ as in (7) by sending it to $\phi \circ g^{-1}$.

The problem with using (9) as a definition is that if G is a reductive algebraic group acting on a projective variety Y, it is not always the case that there exists a geometric quotient Y/G (i.e. a scheme whose closed points are in one-to-one correspondence with G-orbits of Y). However, geometric invariant theory ([13]) allows one to define an open subset $Y^{\text{stable}} \subset Y$ of **stable points**, such that Y^{stable}/G is a geometric quotient. The following is proved, for instance, in [11]:

Theorem 2. *A closed point* (7) *of* $\text{Quot}_{\mathcal{O}_X(-n)^{P(n)},P}$ *is stable under the action of* $GL_{P(n)}$ *from* (9) *if and only if the sheaf* $\mathcal{F}$ *has the property that:*

$$p_{\mathcal{G}}(n) < p_{\mathcal{F}}(n), \quad \textit{for } n \textit{ large enough}$$

for any proper subsheaf $\mathcal{G} \subset \mathcal{F}$*, where the reduced Hilbert polynomial* $p_{\mathcal{F}}(n)$ *is defined as the Hilbert polynomial* $P_{\mathcal{F}}(n)$ *divided by its leading order term.*

Therefore, putting the previous paragraphs together, there is a scheme:

$$\mathcal{M}_P := \Big\{ \mathcal{F} \ \underline{\text{stable}} \text{ coherent sheaf on } X \text{ with Hilbert polynomial } P \Big\} \tag{10}$$

which is defined as the geometric quotient:

$$\mathcal{M}_P = \text{Quot}^{\text{stable}}_{\mathcal{O}_X(-n)^{P(n)},P} / GL_{P(n)} \tag{11}$$

Moreover, [11] prove that under certain numerical hypotheses (specifically, that the coefficients of the Hilbert polynomial $P(n)$ written in the basis $\binom{n+i-1}{i}$ be coprime integers) there exists a universal sheaf $\mathcal{U}$ on $\mathcal{M}_P \times X$. This sheaf is supposed to ensure that the identification (1) is given explicitly by (2), and we note that the universal sheaf is only defined up to tensoring with an arbitrary line bundle pulled back from $\mathcal{M}_P$. We fix such a choice throughout this paper.

1.4 Tangent spaces

From now on, let us restrict to the case of moduli spaces $\mathcal{M}$ of stable sheaves over a smooth projective surface S. Then the Hilbert polynomial of any coherent sheaf is completely determined by 3 invariants: the rank r, and the first and second Chern classes c_1 and c_2 of the sheaf. We will therefore write:

$$\mathcal{M}_{(r,c_1,c_2)} \subset \mathcal{M}$$

for the connected component of $\mathcal{M}$ which parametrizes stable sheaves on S with the invariants r, c_1, c_2. For the remainder of this paper, we will make:

$$\textbf{Assumption A:} \quad \gcd(r, c_1 \cdot \mathcal{O}(1)) = 1 \tag{12}$$

As explained in the last paragraph of the preceding Subsection, Assumption A implies that there exists a universal sheaf on $\mathcal{M} \times S$.

Exercise 1. *Compute the Hilbert polynomial of a coherent sheaf $\mathcal{F}$ on a smooth projective surface S in terms of the invariants r, c_1, c_2 of $\mathcal{F}$, the invariants $c_1(S), c_2(S)$ of the tangent bundle of S, and the first Chern class of the line bundle $\mathcal{O}_S(1)$ (Hint: use the Grothendieck-Hirzebruch-Riemann-Roch theorem).*

The closed points of the scheme $\mathcal{M}$ are $\text{Maps}(\mathbb{C}, \mathcal{M})$, which according to (1) are in one-to-one correspondence to stable coherent sheaves $\mathcal{F}$ on S. As for the tangent space to $\mathcal{M}$ at such a closed point $\mathcal{F}$, it is given by:

$$\text{Tan}_{\mathcal{F}}\mathcal{M} = \left\{ \text{maps Spec } \frac{\mathbb{C}[\varepsilon]}{\varepsilon^2} \xrightarrow{\Psi} \mathcal{M} \text{ which restrict to Spec } \mathbb{C} \xrightarrow{\mathcal{F}} \mathcal{M} \text{ at } \varepsilon = 0 \right\} \tag{13}$$

Under the interpretation (1) of $\text{Maps}(\mathbb{C}[\varepsilon]/\varepsilon^2, \mathcal{M})$, one can prove the following:

Exercise 2. *The vector space* $\text{Tan}_{\mathcal{F}}\mathcal{M}$ *is naturally identified with* $\text{Ext}^1(\mathcal{F}, \mathcal{F})$.

It is well-known that a projective scheme (over an algebraically closed field of characteristic zero) is smooth if and only if all of its tangent spaces have the same dimension. Using this, one can prove:

Exercise 3. *The scheme $\mathcal{M}$ is smooth if the following holds:*

$$\textbf{\textit{Assumption S:}} \quad \begin{cases} \mathcal{K}_S \cong \mathcal{O}_S & \text{or} \\ \mathcal{K}_S \cdot \mathcal{O}(1) < 0 \end{cases} \tag{14}$$

Hint: show that the dimension of the tangent spaces $\text{Ext}^1(\mathcal{F}, \mathcal{F})$ is locally constant, by using the fact that the Euler correspondence:

$$\chi(\mathcal{F}, \mathcal{F}) = \sum_{i=0}^{2} (-1)^i \dim \text{Ext}^i(\mathcal{F}, \mathcal{F})$$

is locally constant, and the fact that stable sheaves $\mathcal{F}$ are simple, i.e. their only automorphisms are scalars (as for $\text{Ext}^2(\mathcal{F}, \mathcal{F})$, you may compute its dimension by using Serre duality on a smooth projective surface).

In fact, one can even compute $\chi(\mathcal{F}, \mathcal{F})$ by using the Grothendieck-Hirzebruch-Riemann-Roch theorem. The exact value will not be important to us, but:

Exercise 4. *Show that (under Assumption S):*

$$\dim \mathcal{M}_{(r,c_1,c_2)} = \text{const} + 2rc_2 \tag{15}$$

where const *is an explicit constant that only depends on* S, r, c_1 *and not on* c_2*.*

1.5 Hecke correspondences - part 1

Fix r and c_1. The moduli space of Hecke correspondences is the locus of pairs:

$$\mathfrak{Z}_1 = \Big\{ \text{pairs } (\mathcal{F}', \mathcal{F}) \text{ s.t. } \mathcal{F}' \subset \mathcal{F} \Big\} \subset \bigsqcup_{c_2 \in \mathbb{Z}} \mathcal{M}_{(r,c_1,c_2+1)} \times \mathcal{M}_{(r,c_1,c_2)} \tag{16}$$

In the setting above, the quotient sheaf $\mathcal{F}/\mathcal{F}'$ has length 1, and must therefore be isomorphic to $\mathbb{C}_x$ for some closed point $x \in S$. If this happens, we will use the notation $\mathcal{F}' \subset_x \mathcal{F}$. We conclude that there exist three maps:

$$\begin{array}{ccccc} & & \mathfrak{Z}_1 & & \\ & \swarrow^{p_+} & \downarrow{\scriptstyle p_S} & \searrow^{p_-} & \\ \mathcal{M} & & S & & \mathcal{M} \end{array} \qquad \begin{array}{ccccc} & & (\mathcal{F}' \subset_x \mathcal{F}) & & \\ & \swarrow^{p_+} & \downarrow{\scriptstyle p_S} & \searrow^{p_-} & \\ \mathcal{F}' & & x & & \mathcal{F} \end{array} \tag{17}$$

It is not hard to see that the maps p_+, p_-, p_S are all proper. In fact, we have the following explicit fact, which also describes the scheme structure of $\mathfrak{Z}_1$:

Exercise 5. *The scheme* $\mathfrak{Z}_1$ *is the projectivization of a universal sheaf:*

$$\begin{array}{c} \mathcal{U} \\ \vdots \\ \downarrow \\ \mathcal{M} \times S \end{array} \tag{18}$$

in the sense that $\mathbb{P}_{\mathcal{M}\times S}(\mathcal{U}) \cong \mathfrak{Z}_1 \xrightarrow{p_- \times p_S} \mathcal{M} \times S$*.*

By definition, the projectivization of $\mathcal{U}$ is:

$$\mathbb{P}_{\mathcal{M}\times S}(\mathcal{U}) = \text{Proj}_{\mathcal{M}\times S}\left(\text{Sym}^*(\mathcal{U})\right) \tag{19}$$

and it comes endowed with a tautological line bundle, denoted by $\mathcal{O}(1)$, and with a map $\rho : \mathbb{P}_{\mathcal{M}\times S}(\mathcal{U}) \to \mathcal{M} \times S$. The scheme (19) is completely determined by the fact that maps $\Phi : T \to \mathbb{P}_{\mathcal{M}\times S}(\mathcal{U})$ are in one-to-one correspondence with triples consisting of the following: a map $\phi : T \to \mathcal{M} \times S$, a line bundle $\mathcal{L}$ on T (which will be the pull-back of $\mathcal{O}(1)$ under Φ), and a surjective map $\phi^*(\mathcal{U}) \twoheadrightarrow \mathcal{L}$. However, in the case at hand, we can describe (19) a bit more explicitly:

Exercise 6. *There is a short exact sequence on* $\mathcal{M} \times S$*:*

$$0 \to \mathcal{W} \to \mathcal{V} \to \mathcal{U} \to 0 \tag{20}$$

where $\mathcal{V}$ *and* $\mathcal{W}$ *are locally free sheaves on* $\mathcal{M} \times S$ *(see Proposition 2.2 of [18]).*

As a consequence of Exercise 6, we have an embedding:

$$\begin{array}{ccc} \mathfrak{Z}_1 \cong \mathbb{P}_{\mathcal{M}\times S}(\mathcal{U}) & \stackrel{\iota}{\hookrightarrow} & \mathbb{P}_{\mathcal{M}\times S}(\mathcal{V}) \\ & \searrow^{p_- \times p_S} & \downarrow \rho \\ & & \mathcal{M} \times S \end{array} \tag{21}$$

which is very helpful, since $\mathbb{P}_{\mathcal{M}\times S}(\mathcal{V})$ is a projective space bundle over $\mathcal{M} \times S$, hence smooth. Moreover, one can even describe the ideal of the embedding ι above. Try to show that it is equal to the image of the map:

$$\rho^*(\mathcal{W}) \otimes \mathcal{O}(-1) \to \rho^*(\mathcal{V}) \otimes \mathcal{O}(-1) \to \mathcal{O} \tag{22}$$

on $\mathbb{P}_{\mathcal{M}\times S}(\mathcal{V})$. Therefore, we conclude that $\mathfrak{Z}_1$ is cut out by a section of the vector bundle $\rho^*(\mathcal{W}^\vee) \otimes \mathcal{O}(1)$ on the smooth scheme $\mathbb{P}_{\mathcal{M}\times S}(\mathcal{V})$.

Exercise 7. *Under Assumption S,* $\mathfrak{Z}_1$ *is smooth of dimension:*

$$\text{const} + r(c_2 + c_2') + 1$$

where c_2 *and* c_2' *are the locally constant functions on* $\mathfrak{Z}_1 = \{(\mathcal{F}' \subset \mathcal{F})\}$ *which keep track of the second Chern classes of the sheaves* $\mathcal{F}$ *and* $\mathcal{F}'$*, respectively. The number* const *is the same one that appears in* (15)*.*

Exercise 7 is a well-known fact, which was first discovered for Hilbert schemes more than 20 years ago. You may prove it by describing the tangent spaces to $\mathfrak{Z}_1$ in terms of Ext groups (emulating the isomorphism (13) of the previous Subsection), or by looking at Proposition 2.10 of [18]. As a consequence of the dimension estimate in Exercise 7, it follows that the section (22) is regular, and so $\mathfrak{Z}_1$ is regularly embedded in the smooth variety $\mathbb{P}_{\mathcal{M}\times S}(\mathcal{V})$.

1.6 K-theory and derived categories

The schemes $\mathcal{M}$ and $\mathfrak{Z}_1$ will play a major role in what follows, but we must first explain what we wish to do with them. Traditionally, the enumerative geometry of such moduli spaces of sheaves is encoded in their cohomology, but in the present notes we will mostly be concerned with more complicated invariants. First of all, we have their K–theory groups:

$$K_{\mathcal{M}} \quad \text{and} \quad K_{\mathfrak{Z}_1}$$

which are defined as the $\mathbb{Q}$-vector spaces generated by isomorphism classes of locally free sheaves on these schemes, modulo the relation $[\mathcal{F}] = [\mathcal{G}] - [\mathcal{H}]$ whenever we have a short exact sequence of locally free sheaves $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$.

Example 4. *K–theory is always a ring, with respect to direct sum and tensor product of vector bundles. In particular, we have a ring isomorphism:*

$$K_{\mathbb{P}^n} \to \frac{\mathbb{Q}[\xi]}{(1-\xi)^{n+1}}, \qquad \mathcal{O}(1) \mapsto \xi$$

Functoriality means that if $f : X \to Y$, then there should exist homomorphisms:

$$K_X \underset{f^*}{\overset{f_*}{\rightleftarrows}} K_Y$$

called push-forward and pull-back (or direct image and inverse image, respectively). With our definition, the pull-back f^* is well-defined in complete generality, while the push-forward f_* is well-defined when f is proper and Y is smooth.

Remark 1. *There is an alternate definition of K–theory, which uses isomorphism classes of coherent sheaves instead of locally free ones. On a smooth projective scheme, the two notions are equivalent because any coherent sheaf has a finite resolution in terms of locally free sheaves. However, the version of K–theory that uses coherent sheaves has different functoriality properties: the existence of the push-forward f_* only requires $f : X \to Y$ to be a proper morphism (with no restriction on Y), while the pull-back f^* requires f to be an l.c.i. morphism (or at least to satisfy a suitable Tor finiteness condition).*

K–theory is a shadow of a more complicated notion, known as the derived category of perfect complexes, which we will denote by:

$$D_{\mathcal{M}} \quad \text{and} \quad D_{\mathfrak{Z}_1}$$

Specifically, the derived category of a projective variety has objects given by complexes of locally free sheaves and morphisms given by maps of complexes, modulo homotopies, and inverting quasi-isomorphisms (in other words, any map of complexes which induces isomorphisms on cohomology is formally considered to be an isomorphism in the derived category). There is a natural map:

$$\text{Obj } D_X \to K_X$$

which sends a complex of locally free sheaves to the alternating sum of its cohomology groups. Since derived categories have, more or less, the same functoriality properties as K–theory groups, we will not review them here. However, we will compare Example 4 with the following result, due to Beilinson:

Example 5. *Any complex in $D_{\mathbb{P}^n}$ is quasi-isomorphic to a complex of direct sums and homological shifts of the line bundles $\{\mathcal{O}, \mathcal{O}(1), ..., \mathcal{O}(n)\}$. The Koszul complex:*

$$\left[\mathcal{O} \to \mathcal{O}(1)^{\oplus n+1} \to \mathcal{O}(2)^{\oplus \binom{n+1}{2}} \to ... \to \mathcal{O}(n)^{\oplus \binom{n+1}{n}} \to \mathcal{O}(n+1)\right]$$

is exact, hence quasi-isomorphic to 0 in $D_{\mathbb{P}^n}$. This is the categorical version of:

$$(1-\xi)^{n+1} = 1-(n+1)\xi+\binom{n+1}{2}\xi^2-...+(-1)^n\binom{n+1}{n}\xi^n+(-1)^{n+1}\xi^{n+1} = 0$$

which is precisely the relation from Example 4.

2 Representation theory

2.1 Heisenberg algebras and Hilbert schemes

Consider the Hilbert scheme Hilb_d of d points on a smooth projective algebraic surface S. A basic problem is to compute the Betti numbers:

$$b_i(\mathrm{Hilb}_d) = \dim_{\mathbb{Q}} H^i(\mathrm{Hilb}_d, \mathbb{Q})$$

and their generating function $B(\mathrm{Hilb}_d, t) = \sum_{i \geq 0} t^i b_i(\mathrm{Hilb}_d)$. It turns out that these are easier computed if we consider all d from 0 to ∞ together, as was revealed in the following formula (due to Ellingsrud and Strømme [3] for $S = \mathbb{A}^2$ and then to Göttsche [8] in general):

$$\sum_{d=0}^{\infty} q^d B(\mathrm{Hilb}_d, t) = \prod_{i=1}^{\infty} \frac{(1+t^{2i-1}q^i)^{b_1(S)}(1+t^{2i+1}q^i)^{b_3(S)}}{(1-t^{2i-2}q^i)^{b_0(S)}(1-t^{2i}q^i)^{b_2(S)}(1-t^{2i+2}q^i)^{b_4(S)}} \qquad (23)$$

The reason for the formula above was explained, independently, by Grojnowski [9] and Nakajima [14]. To summarize, they introduced an action of a Heisenberg algebra (to be defined) associated to the surface S on the cohomology group:

$$H = \bigoplus_{d=0}^{\infty} H_d, \qquad \text{where} \qquad H_d = H^*(\mathrm{Hilb}_d, \mathbb{Q}) \qquad (24)$$

Since the Betti numbers are just the graded dimensions of H, formula (23) becomes a simple fact about characters of representations of the Heisenberg algebra. The immediate conclusion is that the representation theory behind the action $\mathrm{Heis} \curvearrowright H$ can help one prove numerical properties of H.

Definition 4. *The* ***Heisenberg*** *algebra* Heis *is generated by infinitely many symbols* $\{a_n\}_{n \in \mathbb{Z} \backslash 0}$ *modulo the relation:*

$$[a_n, a_m] = \delta_{n+m}^0 n \qquad (25)$$

Let us now describe the way the Heisenberg algebra of Definition 4 acts on the cohomology groups (24). It is not as straightforward as having a ring homomorphism $\mathrm{Heis} \to \mathrm{End}(H)$, but it is morally very close. To this end, let us recall Nakajima's formulation of the action from [14]. Consider the closed subset:

$$\mathrm{Hilb}_{d,d+n} = \Big\{(I, I', x) \text{ such that } I' \subset_x I\Big\} \in \mathrm{Hilb}_d \times \mathrm{Hilb}_{d+n} \times S \qquad (26)$$

(recall that $I' \subset_x I$ means that the quotient I/I' is a finite length sheaf, specifically length n, supported at the closed point x). We have three natural maps:

$$\begin{array}{ccccc} & & \mathrm{Hilb}_{d,d+n} & & \\ & \swarrow^{p_-} & \downarrow^{p_S} & \searrow^{p_+} & \\ \mathrm{Hilb}_d & & S & & \mathrm{Hilb}_{d+n} \end{array} \qquad (27)$$

While the schemes Hilb_d and S are smooth, $\text{Hilb}_{d,d+n}$ are not. However, the maps $p_\pm$, p_S are proper, and therefore the following operators are well-defined:

$$H_d \xrightarrow{A_n} H_{d+n} \otimes H_S \qquad A_n = (p_+ \times p_S)_* \circ p_-^* \tag{28}$$

$$H_{d+n} \xrightarrow{A_{-n}} H_d \otimes H_S \qquad A_{-n} = (-1)^{n-1} \cdot (p_- \times p_S)_* \circ p_+^* \tag{29}$$

where $H_S = H^*(S, \mathbb{Q})$. We will use the notation $A_{\pm n}$ for the operators above for all d, so one should better think of $A_{\pm n}$ as operators $H \to H \otimes H_S$. Then the main result of Nakajima and Grojnowski states (in a slightly rephrased form):

Theorem 3. *We have the following equality of operators $H \to H \otimes H_S \otimes H_S$:*

$$[A_n, A_m] = \delta^0_{n+m} n \cdot \text{Id}_H \otimes [\Delta] \tag{30}$$

where in the left hand side we take the difference of the compositions:

$$H \xrightarrow{A_m} H \otimes H_S \xrightarrow{A_n \otimes \text{Id}_S} H \otimes H_S \otimes H_S$$

$$H \xrightarrow{A_n} H \otimes H_S \xrightarrow{A_m \otimes \text{Id}_S} H \otimes H_S \otimes H_S \xrightarrow{\text{Id}_H \otimes \text{swap}} H \otimes H_S \otimes H_S$$

and in the right-hand side we multiply by the Poincaré dual class of the diagonal $\Delta \hookrightarrow S \times S$ in $H^*(S \times S, \mathbb{Q}) = H_S \otimes H_S$. The word "swap" refers to the permutation of the two factors of H_S, and the reason it appears is that we want to ensure that in (30) *the operators A_n, A_m each act in a single tensor factor of $H_S \otimes H_S$.*

We will refer to the datum $A_n : H \to H \otimes H_S$, $n \in \mathbb{Z} \backslash 0$ as an action of the Heisenberg algebra on H, and relation (30) will be a substitute for (25). There are two ways one can think about this: the first is that the ring H_S is like the ring of constants for the operators A_n. The second is that one can obtain actual endomorphisms of H associated to any class $\gamma \in H_S$ by the expressions:

$$H \xrightarrow{A_n^\gamma} H \qquad A_n^\gamma = \int_S \gamma \cdot A_n$$

$$H \xrightarrow{A_{-n}^\gamma} H \qquad A_{-n}^\gamma = \int_S \gamma \cdot A_{-n}$$

where $\int_S : H_S \to \mathbb{Q}$ is the integration of cohomology on S. It is not hard to show that (30) yields the following commutation relation of operators $H \to H$:

$$[A_n^\gamma, A_m^{\gamma'}] = \delta^0_{n+m} n \int_S \gamma\gamma' \cdot \text{Id}_H$$

for any classes $\gamma, \gamma' \in H_S$. The relation above is merely a rescaled version of relation (25), and it shows that each operator $A_n : H \to H \otimes H_S$ defined by Nakajima and Grojnowski entails the same information as a family of endomorphisms of H indexed by the cohomology group of the surface S itself.

2.2 Going forward

Baranovsky [1] generalized Theorem 3 to the setup where Hilbert schemes are replaced by the moduli spaces of stable sheaves:

$$\mathcal{M} = \bigsqcup_{c_2 \in \mathbb{Z}} \mathcal{M}_{(r,c_1,c_2)}$$

from Subsection 1.4 (for fixed r, c_1 and with Assumption S in effect). A different generalization entails going from cohomology to K–theory groups, and this is a bit more subtle. The first naive guess is that one should define operators:

$$K_{\mathcal{M}} \xrightarrow{A_n} K_{\mathcal{M}} \otimes K_S$$

for all $n \in \mathbb{Z} \backslash 0$ which satisfy the following natural deformation of relation (30):

$$[A_n, A_m] = \delta_{n+m}^0 \frac{1 - q^{rn}}{1 - q} \cdot \mathrm{Id}_{K_{\mathcal{M}}} \otimes [\Delta] \tag{31}$$

where q is some invertible parameter (the reason for the appearance of r in the exponent is representation-theoretic, in that K–theory groups of moduli spaces of rank r sheaves yield level r representations of Heisenberg algebras). However, we have already said that the "ground ring" should be K_S, so the parameter q should be an invertible element of K_S (it will later turn out that q is the K–theory class of the canonical line bundle of S) and relation (31) should read:

$$[A_n, A_m] = \delta_{n+m}^0 \mathrm{Id}_{K_{\mathcal{M}}} \otimes \Delta_* \left(\frac{1 - q^{rn}}{1 - q} \right) \tag{32}$$

But even this form of the relation is wrong, mostly because the Künneth formula does not hold (in general) in K–theory: $K_{\mathcal{M} \times S} \ncong K_{\mathcal{M}} \otimes K_S$. Therefore, the operators we seek should actually be:

$$K_{\mathcal{M}} \xrightarrow{A_n} K_{\mathcal{M} \times S} \tag{33}$$

and they must satisfy the following equality of operators $K_{\mathcal{M}} \to K_{\mathcal{M} \times S \times S}$:

$$[A_n, A_m] = \delta_{n+m}^0 \Delta_* \left(\frac{1 - q^{rn}}{1 - q} \cdot \mathrm{proj}^* \right) \tag{34}$$

where $\mathrm{proj} : \mathcal{M} \times S \to \mathcal{M}$ is the natural projection. So you may ask whether the analogues of the operators (28) and (29) in K–theory will do the trick. The answer is no, because the pull-back maps $p_\pm^*$ from (27) are not the right objects to study in K–theory. This is a consequence of the fact that the schemes $\mathrm{Hilb}_{d,d+n}$ are very singular for $n \geq 2$, and even if the pull-back maps $p_\pm^*$ were defined, then it is not clear what the structure sheaf of $\mathrm{Hilb}_{d,d+n}$ should be replaced with in K–theory, in order to give rise to the desired operators (33).

2.3 Framed sheaves on $\mathbb{A}^2$

We will now recall the construction of Schiffmann and Vasserot, which generalize the Heisenberg algebra action in the case when $S = \mathbb{A}^2$, in the setting of equivariant K–theory (with respect to the action of the standard torus $\mathbb{C}^* \times \mathbb{C}^* \curvearrowright \mathbb{A}^2$). Historically, this work is based on the computation of K–theoretic Hall algebras by Ginzburg and Vasserot [24], generalized by Varagnolo and Vasserot [25], and then by Nakajima [15] to the general setting of quiver varieties.

First of all, since the definition of moduli spaces in the previous Section applies to projective surfaces, we must be careful in defining the moduli space $\mathcal{M}$ when $S = \mathbb{A}^2$. The correct definition is the moduli space of framed sheaves on $\mathbb{P}^2$:

$$\mathcal{M} = \left\{ \mathcal{F} \text{ rank } r \text{ torsion-free sheaf on } \mathbb{P}^2, \mathcal{F}|_\infty \stackrel{\phi}{\cong} \mathcal{O}_\infty^{\oplus r} \right\}$$

where $\infty \subset \mathbb{P}^2$ denotes the divisor at infinity. The space $\mathcal{M}$ is a quasi-projective variety, and we will denote its $\mathbb{C}^* \times \mathbb{C}^*$ equivariant K–theory group by $K_{\mathcal{M}}$. One can define the scheme $\mathfrak{Z}_1$ as in Subsection 1.5 and note that it is still smooth. There is a natural line bundle:

$$\begin{array}{c}\mathcal{L} \\ \downarrow \\ \mathfrak{Z}_1\end{array}$$

whose fiber over a closed point $\{(\mathcal{F}' \subset_x \mathcal{F})\}$ is the one-dimensional space $\mathcal{F}_x/\mathcal{F}'_x$. The maps $p_\pm$ of (17) are still well-defined, and they allow us to define operators:

$$K_{\mathcal{M}} \xrightarrow{E_k} K_{\mathcal{M}}, \qquad E_k = p_{+*}\left(\mathcal{L}^{\otimes k} \cdot p_-^*\right) \tag{35}$$

for all $k \in \mathbb{Z}$. Note that the map $p_-^* : K_{\mathcal{M}} \to K_{\mathfrak{Z}_1}$ is well-defined in virtue of the smoothness of the spaces $\mathfrak{Z}_1$ and $\mathcal{M}$. The following is the main result of [24]:

Theorem 4. *The operators* (35) *satisfy the relations in the elliptic Hall algebra.*

To be more precise, [24] define an extra family of operators f_k defined by replacing the signs $+$ and $-$ in (35), as well as a family of multiplication operators h_k, and they show that the three families of operators e_k, f_k, h_k generate the double elliptic Hall algebra. We will henceforth focus only on the algebra generated by the operators $\{e_k\}_{k\in\mathbb{Z}}$ in order to keep things simple, and we will find inside this algebra the positive half of Heis (i.e. the operators (33) for $n > 0$).

Remark 2. *Theorem 4 was obtained simultaneously by Feigin and Tsymbaliuk in [4], by using the Ding-Iohara-Miki algebra instead of the elliptic Hall algebra (Schiffmann showed in [23] that the two algebras are isomorphic). We choose to follow the presentation in terms of the elliptic Hall algebra for two important and inter-related reasons: the generators of the elliptic Hall algebra are suitable for categorification and geometry, and the Heisenberg operators* (33) *can be more explicitly described in terms of the elliptic Hall algebra than in terms of the isomorphic Ding-Iohara-Miki algebra.*

2.4 The elliptic Hall algebra

The elliptic Hall algebra $\mathcal{E}$ was defined by Burban and Schiffmann in [2], as a formal model for part of the Hall algebra of the category of coherent sheaves on an elliptic curve over a finite field. The two parameters q_1 and q_2 over which the elliptic Hall algebra are defined play the roles of Frobenius eigenvalues.

Definition 5. *Write $q = q_1 q_2$. The **elliptic Hall algebra** $\mathcal{E}$ is the $\mathbb{Q}(q_1, q_2)$ algebra generated by symbols $\{a_{n,k}\}_{n\in\mathbb{N},k\in\mathbb{Z}}$ modulo relations* (36) *and* (37)*:*

$$[a_{n,k}, a_{n',k'}] = 0 \tag{36}$$

if $nk' - n'k = 0$, and:

$$[a_{n,k}, a_{n',k'}] = (1-q_1)(1-q_2)\frac{b_{n+n',k+k'}}{1-q^{-1}} \tag{37}$$

if $nk'-n'k = s$ *and* $\{s,1,1\} = \{\gcd(n,k), \gcd(n',k'), \gcd(n+n',k+k')\}$*, where:*

$$1 + \sum_{s=1}^{\infty} \frac{b_{n_0 s, k_0 s}}{x^s} = \exp\left(\sum_{s=1}^{\infty} \frac{a_{n_0 s, k_0 s}(1-q^{-s})}{s x^s}\right)$$

for any coprime n_0, k_0*, where we write* $q = q_1 q_2$*.*

In full generality, the elliptic Hall algebra defined in [2] has generators $a_{n,k}$ for all $(n,k) \in \mathbb{Z}^2 \backslash (0,0)$, and relation (36) is replaced by a deformed Heisenberg algebra relation between the elements $\{a_{ns,ks}\}_{s\in\mathbb{Z}\backslash 0}$, for any coprime n,k. We refer the reader to Section 2 of [19], where the relations in $\mathcal{E}$ are recalled in our notation. However, we need to know two things about this algebra:

- the operators $\{e_k\}_{k\in\mathbb{Z}}$ of (35) will play the role of $a_{1,k}$
- the Heisenberg operators $\{A_n\}_{n\in\mathbb{N}}$ of (34) will play the role of $a_{n,0}$

Therefore, the elliptic Hall algebra contains all the geometric operators we have discussed so far, and more. Therefore, the next natural step is to identify the geometric counterparts of the general operators $a_{n,k}$ (which were first discovered in [17] for $S = \mathbb{A}^2$), but the way to do so will require us to introduce the shuffle algebra presentation of the elliptic Hall algebra.

2.5 The shuffle algebra

Shuffle algebras first arose in the context of Lie theory and quantum groups in the work of Feigin and Odesskii [5]. Various instances of this construction have appeared since, and the one we will mostly be concerned with is the following:

Definition 6. *Let* $\zeta(x) = \frac{(1-q_1 x)(1-q_2 x)}{(1-x)(1-qx)}$*. Consider the* $\mathbb{Q}(q_1,q_2)$*-vector space:*

$$\bigoplus_{n=0}^{\infty} \mathbb{Q}(q_1,q_2)(z_1,...,z_n)^{\text{Sym}} \tag{38}$$

endowed with the following shuffle product for any $f(x_1,...,x_n)$ *and* $g(x_1,...,x_m)$*:*

$$f * g = \text{Sym}\left[f(x_1,...,x_n) g(x_{n+1},...,x_{n+m}) \prod_{n+1\le j\le n+m}^{1\le i\le n} \zeta\left(\frac{x_i}{x_j}\right)\right] \tag{39}$$

where Sym *always refers to symmetrization with respect to all* z *variables. Then the* ***shuffle algebra*** $\mathcal{S}$ *is defined as the* $\mathbb{Q}(q_1,q_2)$*-subalgebra of* (38) *generated by the elements* $\{z_1^k\}_{k\in\mathbb{Z}}$ *in the* $n=1$ *direct summand.*

It was observed in [24] that the map $\mathcal{E} \xrightarrow{\Upsilon} \mathcal{S}$ given by $\Upsilon(a_{1,k}) = z_1^k$ is an isomorphism. The images of the generators $a_{n,k}$ under the isomorphism Υ were worked out in [16], where it was shown that:

$$\Upsilon(a_{n,k}) = \text{Sym}\left[\frac{\prod_{i=1}^{n} z_i^{\lceil \frac{ki}{n} \rceil - \lceil \frac{k(i-1)}{n} \rceil + \delta_i^n - \delta_i^0}}{\left(1 - \frac{qz_2}{z_1}\right) \dots \left(1 - \frac{qz_n}{z_{n-1}}\right)} \prod_{1 \leq i < j \leq n} \zeta\left(\frac{z_i}{z_j}\right)\right.$$

$$\left.\left(1 + \frac{qz_{a(s-1)+1}}{z_{a(s-1)}} + \frac{q^2 z_{a(s-1)+1} z_{a(s-2)+1}}{z_{a(s-1)} z_{a(s-2)}} + \dots + \frac{q^{s-1} z_{a(s-1)+1} \dots z_{a+1}}{z_{a(s-1)} \dots z_a}\right)\right] \quad (40)$$

where $s = \gcd(n, k)$ and $a = n/s$. Note that it is not obvious that the elements (40) are in the shuffle algebra, and the way [16] proves this fact is by showing that $\mathcal{S}$ coincides with the linear subspace of (38) generated by rational functions:

$$\frac{r(z_1, \dots, z_n)}{\prod_{1 \leq i \neq j \leq n} (z_i - qz_j)}$$

where r goes over all symmetric Laurent polynomials that vanish at $\{z_1, z_2, z_3\} = \{1, q_1, q\}$ and at $\{z_1, z_2, z_3\} = \{1, q_2, q\}$. These vanishing properties are called the **wheel conditions**, following those initially introduced in [5].

Formula (40) shows the importance of considering the following elements of $\mathcal{E}$:

$$e_{k_1,\dots,k_n} = \Upsilon^{-1}\left(\text{Sym}\left[\frac{z_1^{k_1} \dots z_n^{k_n}}{\left(1 - \frac{qz_2}{z_1}\right) \dots \left(1 - \frac{qz_n}{z_{n-1}}\right)} \prod_{1 \leq i < j \leq n} \zeta\left(\frac{z_i}{z_j}\right)\right]\right) \quad (41)$$

for any $k_1, \dots, k_n \in \mathbb{Z}$.

Exercise 8. *Show that the right-hand side of* (41) *lies in* $\mathcal{S}$ *by showing that it satisfies the wheel conditions that we discussed previously.*

Exercise 9. *Prove the following commutation relations, for all* $d, k_1, \dots, k_n \in \mathbb{Z}$*:*

$$[e_{k_1,\dots,k_n}, e_d] = (1 - q_1)(1 - q_2) \sum_{i=1}^{n} \begin{cases} \sum_{k_i \leq a < d} e_{k_1,\dots,k_{i-1},a,k_i+d-a,k_{i+1},\dots,k_n} & \text{if } d > k_i \\ -\sum_{d \leq a < k_i} e_{k_1,\dots,k_{i-1},a,k_i+d-a,k_{i+1},\dots,k_n} & \text{if } d < k_i \end{cases} \quad (42)$$

There is no summand in the right-hand side corresponding to $k_i = d$*. You may prove* (42) *by expressing it as an equality of rational functions in the shuffle algebra* $\mathcal{S}$*, which you may then prove explicitly (it is not hard, but also not immediate, so try the cases* $n \in \{1, 2\}$ *first).*

It is clear from relations (36) and (37) that the elements $e_k = a_{1,k}$ generate the algebra $\mathcal{E}$, since any $a_{n,k}$ can be written in terms of sums and products of e_k's. Using the main result of [23], one may show that (42) control all relations among the generators $e_k \in \mathcal{E}$. In fact, these relations are over-determined, but we like them because they allow us to express linear combinations of $e_{k_1,\dots,k_n}$ as explicit commutators of e_k's. Therefore, if you have an action of $\mathcal{E}$ where you know how the e_k act, and you wish to prove that the operators $e_{k_1,\dots,k_n}$ act by some formula $(*)$, all you need to do is prove that formula $(*)$ satisfies (42).

2.6 The action of $\mathcal{E}$ on $K_\mathcal{M}$

We will now apply the philosophy in the previous paragraph to the setting of the K–theory group of the moduli space $\mathcal{M}$ of sheaves on a smooth projective surface S (with fixed rank r and first Chern class c_1). As we have seen in Subsection 2.2, the way to go is to define operators:

$$K_\mathcal{M} \xrightarrow{E_{k_1,\dots,k_n}} K_{\mathcal{M}\times S} \tag{43}$$

for all $k_1, ..., k_n \in \mathbb{Z}$ which satisfy the following analogue of relation (42):

$$[E_{k_1,\dots,k_n}, E_d] =$$
$$= \Delta_* \left(\sum_{i=1}^n \begin{cases} \sum_{k_i \le a < d} E_{k_1,\dots,k_{i-1},a,k_i+d-a,k_{i+1},\dots,k_n} & \text{if } d > k_i \\ -\sum_{d \le a < k_i} E_{k_1,\dots,k_{i-1},a,k_i+d-a,k_{i+1},\dots,k_n} & \text{if } d < k_i \end{cases} \right) \tag{44}$$

as operators $K_\mathcal{M} \to K_{\mathcal{M}\times S\times S}$. The left-hand side is defined as in Theorem 3, taking care that each of the operators $E_{k_1,\dots,k_n}$ and E_d acts in one and the same factor of $S \times S$. The reason why Δ_* is the natural substitute for $(1-q_1)(1-q_2)$ from (42) is that the K–theory class of the diagonal $\Delta \hookrightarrow \mathbb{A}^2 \times \mathbb{A}^2$ is equal to 0 in non-equivariant K–theory, but it is equal to $(1-q_1)(1-q_2)$ equivariantly.

Theorem 5. *There exist operators* (43) *satisfying* (44)*, with E_k given by* (35)*.*

In particular, the operators $E_{0,\dots,0}$ all commute with each other, and they will give rise to the K–theoretic version of the positive half of the Heisenberg algebra from Subsection 2.1. It is possible to extend Theorem 5 to the double of all algebras involved (thus yielding the full Heisenberg) and details can be found in [20].

Given operators $\lambda, \mu : K_\mathcal{M} \to K_{\mathcal{M}\times S}$, let us define the following operations:

$$\lambda\mu|_\Delta = \text{composition } \left\{ K_\mathcal{M} \xrightarrow{\mu} K_{\mathcal{M}\times S} \xrightarrow{\lambda \boxtimes \text{Id}_S} K_{\mathcal{M}\times S\times S} \xrightarrow{\text{Id}_\mathcal{M} \boxtimes \Delta^*} K_{\mathcal{M}\times S} \right\}$$

$$[\lambda, \mu]_{\text{red}} = \nu : K_\mathcal{M} \to K_{\mathcal{M}\times S} \qquad \text{if } \nu \text{ is such that} \qquad [\lambda, \mu] = \Delta_*(\nu)$$

Note that if ν as above exists, it is unique because the map Δ_* is injective (it has a left inverse, i.e. the projection $S \times S \to S$ to one of the factors).

Exercise 10. *For arbitrary $\lambda, \mu, \nu : K_\mathcal{M} \to K_{\mathcal{M}\times S}$, prove the following versions of associativity, the Leibniz rule, and the Jacobi identity, respectively:*

$$(\lambda\mu|_\Delta)\nu|_\Delta = \lambda(\mu\nu|_\Delta)|_\Delta$$
$$[\lambda, \mu\nu|_\Delta]_{\text{red}} = [\lambda, \mu]_{\text{red}}\nu|_\Delta + \mu[\lambda, \nu]_{\text{red}}|_\Delta$$
$$[\lambda, [\mu, \nu]_{\text{red}}]_{\text{red}} + [\mu, [\nu, \lambda]_{\text{red}}]_{\text{red}} + [\nu, [\lambda, \mu]_{\text{red}}]_{\text{red}} = 0$$

Theorem 5 is an explicit way of saying that the operators (43) give rise to an action of the algebra $\mathcal{E}$ on $K_\mathcal{M}$ in the sense that there exists a linear map:

$$\Phi : \mathcal{E} \to \text{Hom}(K_\mathcal{M}, K_{\mathcal{M}\times S}), \qquad \Phi(e_{k_1,\dots,k_n}) = E_{k_1,\dots,k_n}$$

satisfying the following properties for any $x, y \in \mathcal{E}$:

$$\Phi(xy) = \Phi(x)\Phi(y)|_\Delta \tag{45}$$

$$\Phi\left(\frac{[x,y]}{(1-q_1)(1-q_2)}\right) = [\Phi(x), \Phi(y)]_{\text{red}} \tag{46}$$

The parameters q_1 and q_2 act on K_S as multiplication with the Chern roots of the cotangent bundle Ω^1_S. The reason why the left-hand side of (46) makes sense is that for any x, y which are sums of products of the generators $a_{n,k}$ of $\mathcal{E}$, relations (36) and (37) imply that $[x,y]$ is a multiple of $(1-q_1)(1-q_2)$.

3 Proving the main theorem

3.1 Hecke correspondences - part 2

As we have seen, the operators E_k of (43) are defined by using the correspondence $\mathfrak{Z}_1$ and the line bundle $\mathcal{L}$ on it, as in (35). To define the operators $E_{k_1,\dots,k_n}$ in general, we will need to kick up a notch the Hecke correspondences from Subsection 1.5, and therefore we will recycle a lot of the notation therein. Thus, $\mathcal{M}$ is still the moduli space of stable sheaves on a smooth projective surface S with fixed r and c_1, satisfying Assumptions A and S from Subsection 1.4. Consider:

$$\mathfrak{Z}_2 = \Big\{(\mathcal{F}'' \subset \mathcal{F}' \subset \mathcal{F})\Big\} \subset \bigsqcup_{c_2 \in \mathbb{Z}} \mathcal{M}_{(r,c_1,c_2+2)} \times \mathcal{M}_{(r,c_1,c_2+1)} \times \mathcal{M}_{(r,c_1,c_2)} \tag{47}$$

We will denote the support points of flags as above by $x, y \in S$, so that the closed points of $\mathfrak{Z}_2$ take the form $(\mathcal{F}'' \subset_x \mathcal{F}' \subset_y \mathcal{F})$. Consider:

$$\mathfrak{Z}_2 \supset \mathfrak{Z}_2^\bullet = \Big\{(\mathcal{F}'' \subset_x \mathcal{F}' \subset_x \mathcal{F}), x \in S\Big\}$$

In other words, $\mathfrak{Z}_2^\bullet$ is the closed subscheme of $\mathfrak{Z}_2$ given by the condition that the two support points coincide. These two schemes come endowed with maps:

$$\begin{array}{ccccc} & & \mathfrak{Z}_2 & & \\ & \swarrow^{\pi_+} & & \searrow^{\pi_-} & \\ \mathfrak{Z}_1 & & & & \mathfrak{Z}_1 \end{array} \qquad \begin{array}{ccccc} & & \mathfrak{Z}_2^\bullet & & \\ & \swarrow^{\pi_+^\bullet} & & \searrow^{\pi_-^\bullet} & \\ \mathfrak{Z}_1 & & & & \mathfrak{Z}_1 \end{array} \tag{48}$$

where:

$$\pi_+(\mathcal{F}'' \subset \mathcal{F}' \subset \mathcal{F}) = (\mathcal{F}'' \subset \mathcal{F}'), \qquad \pi_-(\mathcal{F}'' \subset \mathcal{F}' \subset \mathcal{F}) = (\mathcal{F}' \subset \mathcal{F})$$

The maps $\pi_\pm^\bullet$ are given by the same formulas as $\pi_\pm$. The maps $\pi_\pm$ and $\pi_\pm^\bullet$ can be realized as explicit projectivizations, as in (19). To see this, let us consider the following coherent sheaves on $\mathfrak{Z}_1 \times S$:

$$\mathcal{W}' = (p_+ \times \text{Id}_S)^*(\mathcal{W}),\ \mathcal{V}' = (p_+ \times \text{Id}_S)^*(\mathcal{V}),\ \mathcal{U}' = (p_+ \times \text{Id}_S)^*(\mathcal{U}) \quad \text{on } \mathfrak{Z}_1 \times S$$

$$\mathcal{W}'^\bullet = (p_+ \times p_S)^*(\mathcal{W}),\ \mathcal{V}'^\bullet = (p_+ \times p_S)^*(\mathcal{V}),\ \mathcal{U}'^\bullet = (p_+ \times p_S)^*(\mathcal{U}) \quad \text{on } \mathfrak{Z}_1$$

More explicitly, we have:

$$\mathcal{U}'_{(\mathcal{F}' \subset_y \mathcal{F}'', x)} = \mathcal{F}'|_x \qquad \mathcal{U}'^\bullet_{(\mathcal{F}' \subset_y \mathcal{F}'')} = \mathcal{F}'|_y$$

and $\mathcal{V}', \mathcal{W}', \mathcal{V}'^\bullet, \mathcal{W}'^\bullet$ are described analogously.

Exercise 11. *The scheme* $\mathfrak{Z}_2$ *is the projectivization of* $\mathcal{U}'$*, in the sense that:*

$$\mathbb{P}_{\mathfrak{Z}_1 \times S}(\mathcal{U}') \cong \mathfrak{Z}_2 \xrightarrow{\pi_-} \mathfrak{Z}_1$$

Pull-backs do not preserve short exact sequences in general, because tensor product is not left exact. As a consequence of this phenomenon, it turns out that the short exact sequence (20) yields short exact sequences:

$$0 \to \mathcal{W}' \to \mathcal{V}' \to \mathcal{U}' \to 0 \qquad \text{on } \mathfrak{Z}_1 \times S \tag{49}$$

$$0 \to \frac{\mathcal{W}'^{\bullet}}{\mathcal{L}'^{\bullet}} \to \mathcal{V}'^{\bullet} \to \mathcal{U}'^{\bullet} \to 0 \qquad \text{on } \mathfrak{Z}_1 \tag{50}$$

where $\mathcal{L}'^{\bullet}$ is an explicit line bundle that the interested reader can find in Proposition 2.18 of [20]. Therefore, Exercise 11 implies that we have diagrams:

$$\begin{array}{ccc} \mathfrak{Z}_2 \cong \mathbb{P}_{\mathfrak{Z}_1}(\mathcal{U}') & \stackrel{\iota'}{\hookrightarrow} & \mathbb{P}_{\mathfrak{Z}_1 \times S}(\mathcal{V}') \\ & \searrow^{\pi_-} & \downarrow \rho' \\ & & \mathfrak{Z}_1 \times S \end{array} \tag{51}$$

$$\begin{array}{ccc} \mathfrak{Z}_2^{\bullet} \cong \mathbb{P}_{\mathfrak{Z}_1}(\mathcal{U}'^{\bullet}) & \stackrel{\iota'^{\bullet}}{\hookrightarrow} & \mathbb{P}_{\mathfrak{Z}_1}(\mathcal{V}'^{\bullet}) \\ & \searrow^{\pi_-^{\bullet}} & \downarrow \rho'^{\bullet} \\ & & \mathfrak{Z}_1 \end{array} \tag{52}$$

The ideals of the embeddings ι' and $\iota'^{\bullet}$ are the images of the maps:

$$\rho'^{*}(\mathcal{W}') \otimes \mathcal{O}(-1) \to \rho'^{*}(\mathcal{V}') \otimes \mathcal{O}(-1) \to \mathcal{O} \tag{53}$$

$$\rho'^{\bullet *}\left(\frac{\mathcal{W}'^{\bullet}}{\mathcal{L}'^{\bullet}}\right) \otimes \mathcal{O}(-1) \to \rho'^{\bullet *}(\mathcal{V}'^{\bullet}) \otimes \mathcal{O}(-1) \to \mathcal{O} \tag{54}$$

on $\mathbb{P}_{\mathfrak{Z}_1 \times S}(\mathcal{V}')$ and $\mathbb{P}_{\mathfrak{Z}_1}(\mathcal{V}'^{\bullet})$, respectively. The following Proposition, analogous to Exercise 7, implies that the embeddings ι' and $\iota'^{\bullet}$ are regular. In other words, the compositions (53) and (54) are duals of regular sections of vector bundles. The regularity of the latter section would fail if we used $\mathcal{W}'^{\bullet}$ instead of $\mathcal{W}'^{\bullet}/\mathcal{L}'^{\bullet}$.

Proposition 1. *([20]) Under Assumption S,* $\mathfrak{Z}_2$ *and* $\mathfrak{Z}_2^{\bullet}$ *have dimensions:*

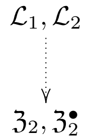
$$\text{const} + r(c_2 + c_2'') + 2 \qquad \textit{and} \qquad \text{const} + r(c_2 + c_2'') + 1$$

respectively, where c_2 *and* c_2'' *are the locally constant functions on the scheme* $\mathfrak{Z}_2 = \{(\mathcal{F}'' \subset \mathcal{F}' \subset \mathcal{F})\}$ *which keep track of the second Chern classes of the sheaves* $\mathcal{F}$ *and* $\mathcal{F}''$*. Moreover,* $\mathfrak{Z}_2$ *is an l.c.i. scheme, while* $\mathfrak{Z}_2^{\bullet}$ *is smooth.*

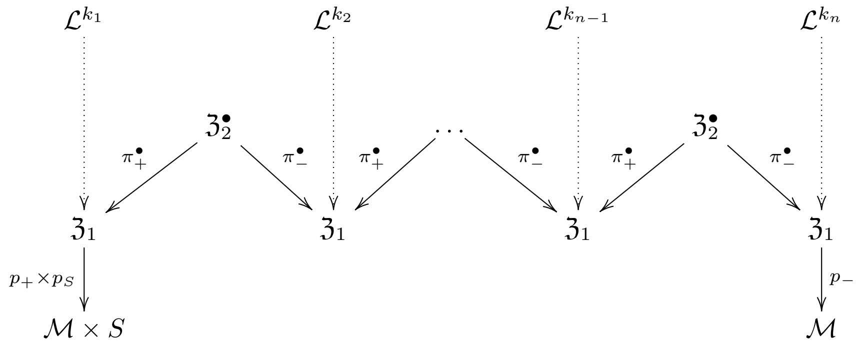
It is also easy to describe the singular locus of $\mathfrak{Z}_2$: it consists of closed points $(\mathcal{F}'' \subset_x \mathcal{F}' \subset_y \mathcal{F})$ where $x = y$ and the quotient $\mathcal{F}/\mathcal{F}''$ is a split length 2 sheaf (so isomorphic to a direct sum of two skyscraper sheaves $\mathbb{C}_x \oplus \mathbb{C}_x$).

3.2 The operators

There are two natural line bundles on the schemes $\mathfrak{Z}_2$ and $\mathfrak{Z}_2^\bullet$, denoted by:



whose fiber over a point $\{(\mathcal{F}'' \subset_x \mathcal{F}' \subset_y \mathcal{F})\}$ are the one-dimensional spaces $\mathcal{F}'_x/\mathcal{F}''_x$, $\mathcal{F}_y/\mathcal{F}'_y$, respectively. The maps of (17) and (48) may be assembled into:



for any $k_1, ..., k_n \in \mathbb{Z}$. The above diagram of smooth schemes, morphisms and line bundles gives rise to an operator:

$$K_{\mathcal{M}} \xrightarrow{E_{k_1,\dots,k_n}} K_{\mathcal{M}\times S}$$

by tracing pull-back and push-forward maps from bottom right to bottom left, and whenever we reach the scheme $\mathfrak{Z}_1$ for the i–th time, we tensor by the line bundle $\mathcal{L}^{k_{n+1-i}}$. In symbols:

$$E_{k_1,\dots,k_n} = (p_+ \times p_S)_* \\ \left(\mathcal{L}^{k_1} \cdot \pi^\bullet_{+*}\pi^{\bullet *}_- \left(\mathcal{L}^{k_2} \cdot \pi^\bullet_{+*}...\pi^{\bullet *}_- \left(\mathcal{L}^{k_{n-1}} \cdot \pi^\bullet_{+*}\pi^{\bullet *}_- \left(\mathcal{L}^{k_n} \cdot p_-^*\right)...\right)\right. \right. \tag{55}$$

and we claim that these are the operators whose existence was stipulated in Theorem 5. Recall that this means that the operators $E_{k_1,\dots,k_n}$ defined as above should satisfy relation (44). In the remainder of this lecture, we will prove the said relation in the case $n = 1$, i.e. we will show that:

$$[E_k, E_d] = \Delta_* \left(\begin{cases} \sum_{k \le a < d} E_{a,k+d-a} & \text{if } d > k \\ -\sum_{d \le a < k} E_{a,k+d-a} & \text{if } d < k \end{cases} \right) \tag{56}$$

The proof of (44) for arbitrary n follows the same principle, although it uses some slightly more complicated geometry and auxiliary spaces.

Remark 3. *Note that even the case $k = d$ of (56), i.e. the relation $[E_k, E_k] = 0$, is non-trivial. The reason for this is that the commutator is defined as the* *difference of the following two compositions, as in Theorem 3:*

$$K_{\mathcal{M}} \xrightarrow{E_k} K_{\mathcal{M}\times S} \xrightarrow{E_k \boxtimes \mathrm{Id}_S} K_{\mathcal{M}\times S\times S}$$

$$K_{\mathcal{M}} \xrightarrow{E_k} K_{\mathcal{M}\times S} \xrightarrow{E_k \boxtimes \mathrm{Id}_S} K_{\mathcal{M}\times S\times S} \xrightarrow{\mathrm{Id}_{K_{\mathcal{M}}} \boxtimes \mathrm{swap}} K_{\mathcal{M}\times S\times S}$$

Recall that "swap" is the permutation of the two factors of $S \times S$, and the reason why we apply it to the second composition is that, in a commutator of the form $[E_k, E_d]$, we wish to ensure that each operator acts in one and the same factor of $K_{S\times S}$. However, the presence of "swap" implies that the two compositions above are not trivially equal to each other. Their equality is proved in (56).

Remark 4. *When $k_1 = ... = k_n = 0$, the composition* (55) *makes sense in cohomology instead of K–theory. In this case, it is not hard to see that the resulting operator $E_{0,...,0}$ is equal to A_n of* (28). *Indeed, this follows from the fact that the former operator is (morally speaking) given by the correspondence:*

$$\Big\{(\mathcal{F}' = \mathcal{F}_0 \subset_x \mathcal{F}_1 \subset_x ... \subset_x \mathcal{F}_{n-1} \subset_x \mathcal{F}_n = \mathcal{F},\ \textit{length } \mathcal{F}_i/\mathcal{F}_{i-1} = 1)\Big\} \qquad (57)$$

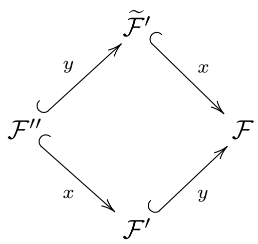
between the moduli spaces parametrizing the sheaves $\mathcal{F}$ and $\mathcal{F}'$, while the rank r generalization of the latter operator ([1]) is given by the correspondence:

$$\Big\{(\mathcal{F}' \subset_x \mathcal{F},\ \textit{length } \mathcal{F}/\mathcal{F}' = n)\Big\} \qquad (58)$$

Since the correspondence (57) *is generically 1-to-1 over the correspondence* (58), *this implies that their fundamental classes give rise to the same operators in cohomology. This argument needs care to be made precise, because the operator $E_{0,...,0}$ is not really given by the fundamental class of* (57), *but by some virtual fundamental class that arises from the composition of operators* (55).

3.3 The moduli space of squares

In order to prove (56), let us consider the space $\mathcal{Y}$ of quadruples of stable sheaves:



(59)

where $x, y \in S$ are arbitrary. There are two maps $\pi^\downarrow, \pi^\uparrow : \mathcal{Y} \to \mathfrak{Z}_2$ which forget the top-most sheaf and the bottom-most sheaf, respectively, and line bundles:

$$\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}'_1, \mathcal{L}'_2 \in \mathrm{Pic}(\mathcal{Y})$$

whose fibers are given by the spaces of sections of the length 1 skyscraper sheaves $\mathcal{F}'/\mathcal{F}''$, $\mathcal{F}/\mathcal{F}'$, $\widetilde{\mathcal{F}}'/\mathcal{F}''$, $\mathcal{F}/\widetilde{\mathcal{F}}'$, respectively. Note that:

$$\mathcal{L}_1\mathcal{L}_2 \cong \mathcal{L}'_1\mathcal{L}'_2 \qquad (60)$$

Proposition 2. *([20]) The scheme $\mathcal{Y}$ is smooth, of the same dimension as $\mathfrak{Z}_2$.*

It is easy to see that the map $\mathcal{Y} \xrightarrow{\pi^\downarrow} \mathfrak{Z}_2$ is surjective. The fiber of this map above a closed point $(\mathcal{F}'' \subset_x \mathcal{F}' \subset_y \mathcal{F}) \in \mathfrak{Z}_2$ consists of a single point unless $x = y$ and $\mathcal{F}/\mathcal{F}''$ is a split length 2 sheaf, in which case the fiber is a copy of $\mathbb{P}^1$ (**Exercise**: prove this). Since the locus where $x = y$ and $\mathcal{F}/\mathcal{F}''$ is precisely the singular locus of $\mathfrak{Z}_2$, it should not be surprising that $\mathcal{Y}$ is a resolution of singularities of $\mathfrak{Z}_2$. The situation is made even nicer by the following.

Proposition 3. *([20]) We have $\pi^\downarrow_*(\mathcal{O}_\mathcal{Y}) = \mathcal{O}_{\mathfrak{Z}_2}$ and $R^i\pi^\downarrow_*(\mathcal{O}_\mathcal{Y}) = 0$ for all $i > 0$.*

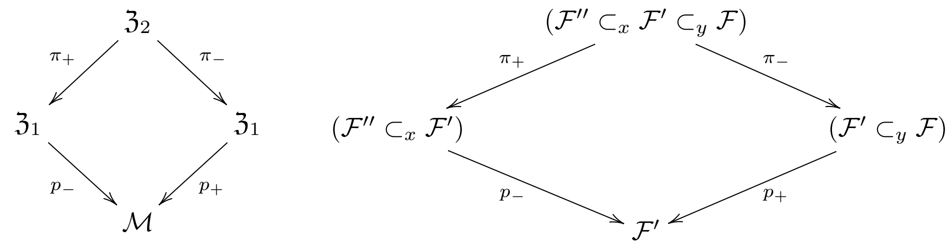
The Proposition above can be proved by embedding $\mathcal{Y}$ into $\mathbb{P}_{\mathfrak{Z}_2}(\mathcal{N})$, where $\mathcal{N}$ is the rank 2 vector bundle on $\mathfrak{Z}_2$ with fibers given $\Gamma(S, \mathcal{F}/\mathcal{F}'')$, and the ideal of this embedding can be explicitly described. As a consequence, one can compute the derived direct images of $\pi^\downarrow$ directly.

Exercise 12. *Find a map of line bundles $\mathcal{L}_1 \xrightarrow{\sigma} \mathcal{L}'_2$ on $\mathcal{Y}$ with zero subscheme:*

$$\mathfrak{Z}_2^\bullet \cong \left\{x = y, \mathcal{F}' = \widetilde{\mathcal{F}}'\right\} \subset \mathcal{Y}$$

3.4 Proof of relation (56)

Consider the diagram:

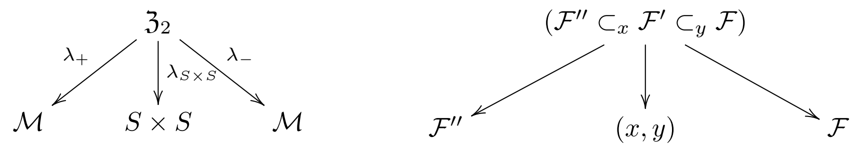


It is clear that the square is Cartesian, but in fact more is true. We know that π_- and p_- are compositions of a regular embedding following by a projective bundle. But comparing (21), (22) with (51), (53), we see that the regular embedding and the projective bundle in question are the same for the two maps π_- and p_-, and this implies that the base change formula holds:

$$p_-^* \circ p_{+*} = \pi_{+*} \circ \pi_-^* : K_{\mathfrak{Z}_1} \to K_{\mathfrak{Z}_1}$$

As a consequence of the formula above, one can show the following:

Exercise 13. *Prove the equality $E_k \circ E_d = (\lambda_+ \times \lambda_{S\times S})_* \Big(\mathcal{L}_1^k \mathcal{L}_2^d \cdot \lambda_-^*\Big)$, where:*



Using Exercise 13 and Proposition 3 implies that:

$$E_k \circ E_d = (\mu_+ \times \mu_{S\times S})_* \Big(\mathcal{L}_1^k \mathcal{L}_2^d \cdot \mu_-^* \Big)$$
$$E_d \circ E_k = (\mu_+ \times \mu_{S\times S})_* \Big({\mathcal{L}'_1}^d {\mathcal{L}'_2}^k \cdot \mu_-^* \Big)$$

where the maps are as follows:

$$\begin{array}{ccccc} & & \mathcal{Y} & & \\ & \overset{\mu_+}{\swarrow} & \downarrow{\scriptstyle \mu_{S\times S}} & \overset{\mu_-}{\searrow} & \\ \mathcal{M} & & S\times S & & \mathcal{M} \end{array} \qquad \begin{array}{ccccc} & & \text{square (59)} & & \\ & \swarrow & \downarrow & \searrow & \\ \mathcal{F}'' & & (x,y) & & \mathcal{F} \end{array}$$

Therefore, assuming $d \geq k$ without loss of generality, we have:

$$[E_k, E_d] = (\mu_+ \times \mu_{S\times S})_* \Big(\Big[\mathcal{L}_1^k \mathcal{L}_2^d - {\mathcal{L}'_1}^d {\mathcal{L}'_2}^k \Big] \cdot \mu_-^* \Big) = \tag{61}$$

$$= (\mu_+ \times \mu_{S\times S})_* \left(\left[1 - \frac{\mathcal{L}_1}{\mathcal{L}'_2} \right] \left[\mathcal{L}_1^k \mathcal{L}_2^d + \mathcal{L}_1^{k+1} \frac{\mathcal{L}_2^d}{\mathcal{L}'_2} + \dots + \mathcal{L}_1^{d-1} \frac{\mathcal{L}_2^d}{{\mathcal{L}'_2}^{d-k-1}} \right] \cdot \mu_-^* \right)$$

where the last equality is a consequence of (60).

Exercise 14. *Show that Exercise 12 implies that:*

$$(\mu_+ \times \mu_{S\times S})_* \left(\left[1 - \frac{\mathcal{L}_1}{\mathcal{L}'_2} \right] \cdot \mathcal{L}_1^e \mathcal{L}_2^f {\mathcal{L}'_1}^g {\mathcal{L}'_2}^h \cdot \mu_-^* \right) = (\nu_+ \times \nu_{S\times S})_* \left(\mathcal{L}_1^{e+g} \mathcal{L}_2^{f+h} \cdot \nu_-^* \right)$$

where the latter maps are as follows:

$$\begin{array}{ccccc} & & \mathfrak{Z}_2^\bullet & & \\ & \overset{\nu_+}{\swarrow} & \downarrow{\scriptstyle \nu_S} & \overset{\nu_-}{\searrow} & \\ \mathcal{M} & & S & & \mathcal{M} \end{array} \qquad \begin{array}{ccccc} & & (\mathcal{F}'' \subset_x \mathcal{F}' \subset_x \mathcal{F}) & & \\ & \swarrow & \downarrow & \searrow & \\ \mathcal{F}'' & & x & & \mathcal{F} \end{array}$$

In terms of the maps (17) *and* (48), *we have* $\nu_\pm = p_\pm \circ \pi_\pm^\bullet$.

Formula (61) and Exercise 14 imply formula (56).

3.5 Toward the derived category

The definition of the operators $E_{k_1,\dots,k_n}$ in (55) immediately generalizes to the derived category (replacing all pull-back and push-forward maps by the corresponding derived inverse and direct image functors), thus yielding functors:

$$D_{\mathcal{M}} \xrightarrow{\widetilde{E}_{k_1,\dots,k_n}} D_{\mathcal{M}\times S}$$

The proof of the previous Subsection immediately shows how to interpret formula (56). Still assuming $d \geq k$, it follows that there exists a natural transformation of functors:

$$\widetilde{E}_d \circ \widetilde{E}_k \to \widetilde{E}_k \circ \widetilde{E}_d$$

whose cone has a filtration with associated graded object:

$$\bigoplus_{a=k}^{d-1} \Delta_* \left(\widetilde{E}_{a,k+d-a}\right)$$

Relation (44) has a similar generalization to the derived category, and the proof uses slightly more complicated spaces instead of $\mathcal{Y}$. The corresponding formula leads one to a categorification $\widetilde{\mathcal{E}}$ of the algebra $\mathcal{E}$, which acts on the derived categories of the moduli spaces $\mathcal{M}$. The complete definition of $\widetilde{\mathcal{E}}$ is still work in progress, but when complete, it should provide a categorification of relations (36)–(37), and in particular a categorification of the Heisenberg algebra (25).

References

[1] Baranovsky V., *Moduli of Sheaves on Surfaces and Action of the Oscillator Algebra*, J. Differential Geom., Volume 55, Number 2 (2000), 193-227

[2] Burban I., Schiffmann O., *On the Hall algebra of an elliptic curve, I*, Duke Math. J. Volume 161, Number 7 (2012), 1171-1231

[3] Ellingsrud G., Strømme S.A., *On the homology of the Hilbert scheme of points in the plane*, Invent. Math. 87 (1987), 343–352

[4] Feigin B., Tsymbaliuk A., *Equivariant K-theory of Hilbert schemes via shuffle algebra*, Kyoto J. Math. Volume 51, Number 4 (2011), 831-854

[5] Feigin B., Odesskii A., *Quantized moduli spaces of the bundles on the elliptic curve and their applications*, Integrable structures of exactly solvable two-dimensional models of quantum field theory (Kiev, 2000), 123-137, NATO Sci. Ser. II Math. Phys. Chem., 35, Kluwer Acad. Publ., Dordrecht, 2001

[6] Ginzburg V., Vasserot E., *Langlands reciprocity for affine quantum groups of type A_n*, Internat. Math. Res. Notices, 3 (1993), 67-85.

[7] Gorsky E., Neguț A., Rasmussen J., *Flag Hilbert schemes, colored projectors and Khovanov-Rozansky homology*, arχiv:1608.07308

[8] Göttsche L., *The Betti numbers of the Hilbert scheme of points on a smooth projective surface*, Math. Ann. 286 (1990), 193–207

[9] Grojnowski I., *Instantons and affine algebras I: the Hilbert scheme and vertex operators*, Math. Res. Letters 3(2), 1995

[10] Hartshorne R., *Algebraic Geometry*, Graduate Texts in Mathematics vol. 52 (1977), Springer-Verlag New York, 978-0-387-90244-9

[11] Huybrechts D., Lehn M., *The geometry of moduli spaces of sheaves*, 2nd edition, Cambridge University Press 2010, ISBN 978-0-521-13420-0

[12] Minets A., *Cohomological Hall algebras for Higgs torsion sheaves, moduli of triples and sheaves on surfaces*, arχiv:1801.01429

[13] Mumford D., Fogarty J., Kirwan F., *Geometric invariant theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], 34 (3rd ed.), 1994, Springer-Verlag

[14] Nakajima H., *Heisenberg algebra and Hilbert schemes of points on projective surfaces*, Ann. of Math. (second series) Vol. 145, No. 2 (Mar., 1997), 379-388

[15] Nakajima H., *Quiver varieties and finite dimensional representations of quantum affine algebras*, J. Amer. Math. Soc. 14 (2001), 145–238

[16] Neguț A., *The shuffle algebra revisited*, Int. Math. Res. Not., Vol. 2014, No. 22, 6242–6275

[17] Neguț A., *Moduli of flags of sheaves and their K-theory*, Algebraic Geometry 2 (1) (2015) 19–43

[18] Neguț A., *Shuffle algebras associated to surfaces*, arχiv:1703.02027

[19] Neguț A., *W-algebras associated to surfaces*, arχiv:1710.03217

[20] Neguț A., *Hecke correspondences for smooth moduli spaces of sheaves*, arχiv:1804.03645

[21] Oblomkov A., Rozansky L., *Knot homology and sheaves on the Hilbert scheme of points on the plane*, Selecta Mathematica 1–104

[22] Sala F., Schiffmann O., *Cohomological Hall algebra of Higgs sheaves on a curve*, arχiv:1801.03482

[23] Schiffmann O., *Drinfeld realization of the elliptic Hall algebra*, Journal of Algebraic Combinatorics, March 2012, Volume 35, Issue 2, 237–262

[24] Schiffmann O., Vasserot E. *The elliptic Hall algebra and the equivariant K-theory of the Hilbert scheme of* $\mathbb{A}^2$, Duke Math. J. Volume 162, Number 2 (2013), 279–366

[25] Varagnolo M., Vasserot E. *On the K-theory of the cyclic quiver variety*, Int. Math. Res. Not. vol 1999, Issue 18, 1005—-1028