Category $\mathcal{O}$, symplectic duality, and the Hikita conjecture

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1 Introduction: The Springer resolution

Let G be a simple algebraic group, $B \subset G$ a Borel subgroup and $Y = G/B$ the flag variety. Thenthe cotangent bundle $\widetilde{X} = T^*Y \cong \{(gB, a) \in Y \times \mathcal{N} | g^{-1}ag \in \mathfrak{b}\}$ has a natural map to the nilpotent cone $X = \mathcal{N}$, called the **Springer resoltion**.

The fibered product $Z = \widetilde{X} \times_X \widetilde{X}$ is called the **Steinberg variety**. It has an alternative description as the union of conormal bundles to G-orbits on $Y \times Y$.

As an example, for $G = SL_2(\mathbb{C})$, $\widetilde{X} = T^*\mathbb{P}^1$, $X = \mathbb{C}^2/\pm 1$ and $Z = T^*\mathbb{P}^1 \cup_{\mathbb{P}^1} \left(\mathbb{P}^1 \times \mathbb{P}^1\right)$.

Fact. *All the irreducible components of Z have the same dimension* $\dim X$.

We also define $\widetilde{X}_+$ to be the union of conormal bundles to the B-orbits in Y inside $\widetilde{X}$, for example, for $Y = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, $\widetilde{X}_+ = \mathbb{C} \cup T_\infty(\mathbb{P}^1)$.

The top Borel-Moore homology $H_{2d}^{BM}(Z)$ of the Steinberg variety has an algebra structure, given by, for $\alpha, \beta \in H_{2d}^{BM}(Z)$

$$\begin{array}{ccccc} & & \widetilde{X} \times_X \widetilde{X} \times_X \widetilde{X} & & \\ & \swarrow^{p_{12}} & \downarrow^{p_{13}} & \searrow^{p_{23}} & \\ Z & & Z & & Z \end{array}$$

$$\alpha * \beta = (p_{13})_* \left((p_{12}^*\alpha) * (p_{23}^*\beta)\right),$$

where p_{ij} are the obvious projection maps.

The top Borel-Moore homology $H_d^{BM}(\widetilde{X}_+)$ becomes a module for this algebra by a similar construction.

Theorem 1.1 (Lusztig, Ginzburg). *We have*

1. $H_{2d}^{BM}(Z) \cong \mathbb{C}[W]$ *as algebras,*
2. $H_d^{BM}(\widetilde{X}_+) \cong \mathbb{C}[W]$ *as modules.*

One might ask: Why go through this complicated construction just to discover the group algebra and the regular representation of W? The answer is that geometric constructions often help with categorizations, where the cohomology groups can be replaced by sheaves.

The group G acts on Y, which leads to a map $U(\mathfrak{g}) \to \mathrm{Diff}(Y) = \Gamma(Y, D_Y)$ to differential operators on Y.

Fact. *The map is surjective.*

Let $U(\mathfrak{g})_0$ be the quotient of $U(\mathfrak{g})$ by the kernel of the map above. Then $U(\mathfrak{g})_0 \cong \mathrm{Diff}_Y$.

Theorem 1.2 (Beilinson-Bernstein). *The map*

$$U(\mathfrak{g})_0 - mod \overset{Loc}{\to} D_Y - mod$$
$$N \mapsto D_Y \otimes_{U(\mathfrak{g})_0} N$$

is an equivalence of categories.

Now that we have a D-module, we can consider its support, or even better, its microlocal support, which will be a cycle on $\widetilde{X}$ as opposed to a cycle on Y.

$$U(\mathfrak{g})_0 - mod \overset{Loc}{\to} D_Y - mod \overset{\text{microlocal supp}}{\to} \text{cycle on } \widetilde{X}$$

What we really want though are cycles on $\widetilde{X}_+$, so we look for a subcategory.

Definition 1.3. *Let $\mathcal{O}_0$ be finitely generated $U(\mathfrak{g})_0$-modules that are locally finite for $U(\mathfrak{b})$.*

Then $\mathcal{O}_0$ maps precisely to cycles on $\widetilde{X}_+$ by microlocal supp $\circ\, Loc$. The point of the local finiteness for $U(\mathfrak{b})$ is that this means that the $U(\mathfrak{b})$-action integrates to aB-action which guarantees that the microlocal support lies in the union of the conormal bundles to B-orbits. Hence we have

$$K(\mathcal{O}_0) \otimes \mathbb{C} \overset{\cong}{\to} H_d^{BM}(\widetilde{X}_+),$$

so $\mathcal{O}_0$ is the categorification we wanted.

Now we want to lift the convolution operators on $H_d^{BM}(\widetilde{X}_+)$ to functors acting on $\mathcal{O}_0$. So we look for suitable bimodules. Similarly to the previous diagram, we consider

$$U(\mathfrak{g})_0 - bimod \overset{Loc}{\to} D_Y \boxtimes D_Y^{opp} \to \text{ cycles on } \widetilde{X} \times \widetilde{X},$$

but again, we want cycles on $Z = \widetilde{X} \times_X \widetilde{X}$, so we look for a subcategory again.

Definition 1.4. *Let HC_0 be finitely generated $U(\mathfrak{g})_0$-bimodules that are locally finite for the adjoint action.*

Then HC_0 maps precisely to cycles on Z by a similar argument as before.

Theorem 1.5. *We have*

1. *HC_0 is a tensor category acting on $\mathcal{O}_0$,*
2. *Taking support intertwines $\otimes^L$ and $*$,*
3. *There are bimodules $\{H_w | w \in W\}$ such that*
   (a) *The functor $\Theta_w : D^b(\mathcal{O}_0) \to D^b(\mathcal{O}_0)$ given by $H_w \otimes^L -$ is an equivalence*
   (b) *$\Theta_w \circ \Theta_{w'} = \Theta_{ww'}$ when $l(w) + l(w') = l(ww')$.*

   so there is an action of the braid group of W on $D^b(\mathcal{O}_0)$ categorifying the W-action on $K(\mathcal{O}_0)$.

2 Symplectic resolutions

The general philosophy is that the above representation theory should happen in the setting of an arbitrary symplectic resolution.

Definition 2.1. *A* ***Conical symplectic resolution*** *is a resolution of singularities $\widetilde{X} \to X$ with a $\mathbb{C}^\times$-action and a symplectic form $\omega \in \Omega^2(\widetilde{X})$ satisfying:*

1. *$\mathbb{C}[X] = \bigoplus_{n \geq 0} \mathbb{C}[X]^n$ with $\mathbb{C}[x]^0 = \mathbb{C}$,*
2. *X is normal,*
3. *$s \cdot \omega = s^2 \omega$*

Some examples of symplectic resolutions are:

1. $\widetilde{X} = T^*G/B, X = \mathcal{N}$,
2. $\widetilde{X} = \text{Hilb}_n(\mathbb{C}^2), X = Sym^n\mathbb{C}^2$
3. Quiver varieties

4. Slices in the affine grassmannian

5. Hypertoric varieties

6. Higgs/Coulomb branches of moduli spaces

We will need an "extra" $\mathbb{C}^\times$-action on $\widetilde{X}$, commuting with S, preserving ω and we also want $|\widetilde{X}^{\mathbb{C}^\times}| < \infty$. For example, any generic cocharacter $\mathbb{C}^\times \to G$ will do for the action of G on G/B. Note: such a $\mathbb{C}^\times$-action might not exist, so this is an additional assumption.

Let $Z = \widetilde{X} \times_X \widetilde{X}$ and $\widetilde{X}_+ = \{p \in \widetilde{X} \,|\, \lim_{t\to 0} t \cdot p \text{ exists}\}$. Then $H_{2d}^{BM}(Z)$ acts on $H_d^{BM}(\widetilde{X}_+)$ as before. We again want to define a category that has $H_d^{BM}(\widetilde{X}_+)$ as its Grothendieck group. We will use quantizations to do this.

Definition 2.2. *A* ***quantization*** *of $\widetilde{X}$ is a sheaf $\mathcal{A}$ of filtered algebras on $\widetilde{X}$ with $[\mathcal{A}^i, \mathcal{A}^j] \subset A^{i+j-2}$ and an isomorphism between $gr\mathcal{A}$ and the structure sheaf of X as graded Poisson algebras.*

Remark 2.3. *The Poisson structure on $gr\mathcal{A}$ is given by lifting, commuting and projecting and has degree -2. The structure sheaf also has a Poisson bracket of degree -2 given by ω.*

Let $A = \Gamma(\widetilde{X}, \mathcal{A})$, so A is filtered with $grA = \mathbb{C}[X]$. One example is $\widetilde{X} = T^*(G/B)$, $\mathcal{A} = \pi^{-1}\mathcal{D}_{G/B}$ given by pulling back D-modules from G/B. Then A is differential operators on G/B and is therefore isomorphic to $U(\mathfrak{g})_0$.

The following should be understood as an analogue of Beilinson-Bernstein localization:

Theorem 2.4 (Braden, Licata, Proudfoot, Webster)**.** *The functor $A - mod \overset{Loc}{\to} \mathcal{A} - mod$ sending $N \mapsto \mathcal{A} \otimes_A N$ is an equivalence of categories for "most" quantizations.*

Recall that we used the support cycles to identify the appropriate subcategory before, we try to replicate this again, we have a map $\mathcal{A} - mod \overset{supp}{\to}$ cycles on $\widetilde{X}$, but we only want to hit $\widetilde{X}_+$.

Definition 2.5. *Let $\mathcal{O}$ be the subcategory of $A - mod$ consisting of objects that are locally finite for the action of $A_+ \subset A$, where A_+ is the sum of the non-negative weight spaces for the extra grading (one should think of this as the analogue of $U(\mathfrak{b})$.*

Proposition 2.6 (Braden, Licata, Proudfoot, Webster)**.** *We have*

$$K(\mathcal{O}) \otimes \mathbb{C} \cong H_d^{BM}(\widetilde{X}_+)$$

Finally we use A-bimodules to categorify the convolution operators. We again have maps

$$\begin{array}{ccccc} A - \text{bimod} & \xrightarrow{Loc} & \mathcal{A} - \text{bimod} & \xrightarrow{supp} & \text{cycles on } \widetilde{X} \times_X \widetilde{X} \\ \uparrow & & & & \uparrow \\ HC & & \longrightarrow & & \text{cycles on } Z \end{array}$$

Theorem 2.7 (Braden, Proudfoot, Webster)**.** *We have*

1. *HC is a tensor category acting on $\mathcal{O}$*

2. *supp intertwines $\otimes$ with $*$*

3. *There are nice HC-bimodules that fit together into a generalized braid group action on $D^b(\mathcal{O})$.*