11 July 2018
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# Category $\mathcal{O}$, symplectic duality, and the Hikita conjecture: Lecture 2

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Recall the context from the previous lecture. We have a conical symplectic resolution $\widetilde{X} \to X$ with a $S = \mathbb{C}^\times$ action:

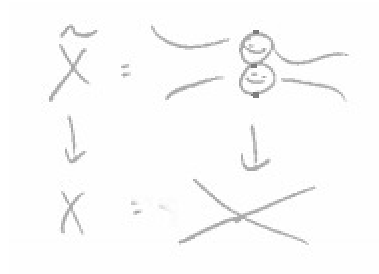
- $\widetilde{X}$ is smooth, symplectic and its symplectic form has weight 2
- $X$ is a normal, affine variety
- $S$ induces a grading on $\mathbb{C}[X] = \oplus_{n\geq 0}\mathbb{C}[X]^n$ such that $\mathbb{C}[X]^0 = \mathbb{C}$ and $\mathbb{C}[X]^1 = 0$.

The condition that $\mathbb{C}[X]^1 = 0$ rules out $\mathbb{C}^2$ with the scalar action (where $x$ and $y$ have weight 1, $dx \wedge dy$ has weight 2) as a trivial factor. Choose a maximal torus $T \subset Aut(X)$, where $Aut(X)$ is the conical symplectic automorphisms of $\widetilde{X}$. We will assume $|\widetilde{X}^T| < \infty$. Choose a generic cocharacter $\mathbb{C}^\times \hookrightarrow T$ as the "extra action". We will denote the grading of $\mathbb{C}[X]$ by $S$ in the top right corner, $\mathbb{C}[X]^n$, and the grading by the "extra action" in the lower right corner, $\mathbb{C}[X]_m$.

**Example 1.** *Consider $\mathbb{Z}/3\mathbb{Z}$ acting on $\mathbb{C}[x, y]$ in the following way. Our generator of $\mathbb{Z}/3\mathbb{Z}$ is a root of unity $\xi$. It acts by $\xi \cdot x = \xi x$, $\xi \cdot y = \xi^{-1} y$. Thus*

$$\begin{aligned} X &= \mathbb{C}^2/(\mathbb{Z}/3\mathbb{Z}) \\ &= \mathrm{Spec}\mathbb{C}[x, y]^{\mathbb{Z}/3\mathbb{Z}} \\ &= \mathrm{Spec}\mathbb{C}[xy, x^3, y^3] \\ &= \mathrm{Spec}\mathbb{C}[a, b, c]/\langle a^3 - bc\rangle \end{aligned}$$

*Our $S$ action scales $\mathbb{C}^2$ with weight $-1$. This implies that $deg(a) = 2$ and $deg(b) = deg(c) = 3$.*



*Here $T = \mathrm{Aut}(X) = \mathbb{C}^\times$ and $t \in T$ acts by $t \cdot a = a$, $t \cdot b = tb$, $t \cdot c = t^{-1}c$. The three marked points make up $\widetilde{X}^T$.*

**Example 2.** *Consider $X^! = \{3 \times 3$ nilpotent matrices of rank $\leq 1\}$ and $\widetilde{X}^! = \{(M, \ell) \mid M \in X^!,\ \ell$ a line containing $im(M)\}$. We have two maps $\pi_1 : \widetilde{X}^! \to X^!$ and $\pi_2 : \widetilde{X}^! \to \mathbb{P}^2$, noting that $\widetilde{X}^! \simeq T^*\mathbb{P}^2$. Note also that $\pi_1$ is usually one-to-one, and only over the zero matrix do we see $\mathbb{P}^2$. The action $S$ scales fibers with weight -2. Here $T^! = (\mathbb{C}^\times)^3/(\mathbb{C}^\times_\Delta)$ in $Aut(X^!) = PGL_3(\mathbb{C})$ is two dimensional. In this case $\left(\widetilde{X}^!\right)^{T^!}$ is the set of coordinate points on $\mathbb{P}^2$.*

The spaces $X$ and $X^!$ are dual to each other in the sense that $\mathcal{O}_X$ is Koszul dual to $\mathcal{O}_{X^!}$. Other examples of dual pairs include:

(1) $T^*(G/B)$ is dual to $T^*(G^L/B^L)$, where $^L$ denotes the Langland dual. (Beilinson-Ginzburg-Soergel)
(2) $\text{Hilb}^n\mathbb{C}^2/\mathbb{C}^2_\Delta$ is self dual. (Rouquier-Shan-Varagnolo-Vasserot)
(3) Quiver varieties are dual to slices in $Gr_G$. (Webster)
(4) Hypertoric varieties are dual to other hypertoric varieties. (Braden-Licata-Proudfoot-Webster)

The dual pair $X$ and $X^!$ from the examples are a special case of both (3) and (4).
**Question:** What is the coordinate ring of $X^T$?

In example 1, $X^T$ is a single point (the three points of $\widetilde{X}^T$ map to the single point of $X^T$). We will see that it has a natural non-reduced scheme structure, and hence its coordinate ring is a finite dimensional algebra.

Suppose group $G$ acts on $X = \text{Spec}R$. So

$$\begin{aligned} p \in X^G &\iff f(p) = f(\sigma p) && \forall f \in R, \sigma \in G \\ &\iff f(p) = (\sigma^{-1} \cdot f)(p) && \forall f \in R, \sigma \in G \end{aligned}$$

Thus we will take

$$X^G = \text{Spec}R/\langle \sigma f - f \mid \sigma \in G, f \in R\rangle$$

as a definition of the fixed point scheme. If $G = \mathbb{C}^\times$, the $\langle \sigma f - f \mid \sigma \in G, f \in R\rangle = \langle$ homogeneous functions of wieght $\neq 0\rangle$. So $X^{\mathbb{C}^\times} = \text{Spec}\mathbb{C}[X]_0/\langle fg \mid wt(f) = -wt(g) \neq 0\rangle$

**Example 1.** *$\mathbb{C}[X] = \mathbb{C}[a,b,c]/\langle a^3 - bc\rangle$, where $wt(a) = 0$, $wt(b) = 1$, and $wt(c) = -1$. Recall from before that $T = \mathbb{C}^\times$. So $\mathbb{C}[X^T] = \mathbb{C}[a]/\langle bc\rangle = \mathbb{C}[a]/\langle a^3\rangle \simeq H^*(\widetilde{X}^!)$.*

**Hikita Conjecture.** If $X$ and $X^!$ are dual, then

$$\mathbb{C}[X^T] \simeq H^*(\widetilde{X}^!).$$

This was proven by Hikita for

- Springer resolution (actually De Concini-Procesi)
- $\text{Hilb}^n\mathbb{C}^2/\mathbb{C}^2_\Delta$
- Finite type A quiver varieties and $Gr_G$ slices
- Hypertoric varieties

Here is a special case of Hikita's conjecture in degree 2:

$$H^2(\widetilde{X}^!) \simeq \mathbb{C}[X]^2_0 \simeq \text{Lie}\,(T) \text{ via the moment map.}$$

Our next aim is a fancier version of Hikita's conjecture in which the coordinate ring is replaced by a quantization. Recall that a quantization of $\widetilde{X}$ is:

- filtered algebra $A$ with $[A^i, A^j] \subset A^{i+j-2}$
- graded Poisson isomorphism $grA \simeq \mathbb{C}[X]$
- "sheafy version"

**Theorem 1** (Bezrukavnikov-Kaledin, Losev)**.** *Quantizations of $\widetilde{X} \xrightarrow{1-1} H^2(\widetilde{X})$.*

**Example 1.** *Considering our main example of $\mathbb{C}[X] = \mathbb{C}[a,b,c]/\langle a^3 - bc\rangle$, we have*

$$\{a,b\} = -b \qquad \{a,c\} = c \qquad \{b,c\} = 3a^2.$$

*Rewrite* $\mathbb{C}[X] = \mathbb{C}[a_1, a_2, a_3, b, c]/\langle a_1a_2a_3 - bc, a_1 - a_2, a_2 - a_3\rangle$*. Define* $A$ *as* $\mathbb{C}\langle a_1, a_2, a_3, b, c\rangle/\langle$ *relations* $\rangle$ *with relations*

$$
\begin{aligned}
[a_i, a_j] &= 0 & bc &= (a_1 + 1)(a_2 + 1)(a_3 + 1) \\
[a_i, b] &= -b & cb &= a_1a_2a_3 \\
[a_i, c] &= c
\end{aligned}
$$

*The center is* $Z(A) = \mathbb{C}[a_1 - a_2, a_2 - a_3] \simeq \mathbb{C}[H^2(\widetilde{X})]$*. Thus the center is a polynomial ring in two variables* $x = a_1 - a_2$ *and* $y = a_2 - a_3$*.*

*Claim: If we set* $x$ *and* $y$ *to complex numbers, then we get a quantization of* $\widetilde{X}$*.*

Sometimes it is convenient to use the Rees algebra construction to turn filtered algebras into algebras over a polynomial ring.

**Definition 1.** $A_{\bar{h}} = Rees(A) = \sum \bar{h}^i A^i \subset \mathbb{C}[\bar{h}] \otimes A$ *is an algebra over* $\mathbb{C}[\bar{h}]$*. Taking* $\bar{h} = 1$ *gets back* $A$*, while taking* $\bar{h} = 0$ *('homogenization') gets* $gr A$*.*

**Example 1.** $A_{\bar{h}} = \mathbb{C}[\bar{h}]\langle a_1, a_2, a_3, b, c\rangle/\langle$ *relations* $\rangle$ *where the relations now are*

$$
\begin{aligned}
[a_i, a_j] &= 0 & bc &= (a_1 + \bar{h})(a_2 + \bar{h})(a_3 + \bar{h}) \\
[a_i, b] &= -\bar{h}b & cb &= a_1a_2a_3 \\
[a_i, c] &= \bar{h}c
\end{aligned}
$$

*This is an algebra over* $\mathbb{C}[H^2(\widetilde{X})] \otimes \mathbb{C}[\bar{h}]$*.*

**Definition 2.** $B_{\bar{h}} = (A_{\bar{h}})_0/\langle fg \mid wt(f) = -wt(g) > 0\rangle$*. Since the algebra is noncommutative, the sign now matters. This is an algebra over* $\mathbb{C}[H^2(\widetilde{X})] \otimes \mathbb{C}[\bar{h}]$*.*

**Remark 1.**

$$(A_{\bar{h}})_- \to (A_{\bar{h}})_0 \twoheadrightarrow B_{\bar{h}}$$

*If* $N$ *is a* $B_{\bar{h}}$*-module, then the "Verma module"* $V = A_{\bar{h}} \otimes_{(A_{\bar{h}})_-} N$ *is a* $A_{\bar{h}}$*-module. If we set* $\bar{h} = 1$ *and specialize at* $h \in H^2(\widetilde{X})$*, then recover a Verma module in* $\mathcal{O}_-$*. So we use* $B_{\bar{h}}$ *to study category* $\mathcal{O}$*.*

**Example 1.** *In our running example,*

$$
\begin{aligned}
B_{\bar{h}} &= \mathbb{C}[\bar{h}, a_1, a_2, a_3]/\langle bc\rangle \\
&= \mathbb{C}[\bar{h}, a_1, a_2, a_3]/\langle (a_1 + \bar{h})(a_2 + \bar{h})(a_3 + \bar{h})\rangle \\
&\simeq H^*_{T^! \times \mathbb{C}^\times}(T^*\mathbb{P}^2)
\end{aligned}
$$

*which is an algebra over* $\mathbb{C}[H^2(\widetilde{X})] \otimes \mathbb{C}[\bar{h}] \simeq \mathbb{C}[\mathrm{Lie}\,(T^!)] \otimes \mathbb{C}[\bar{h}] \simeq H^*_{T^! \times \mathbb{C}^\times}(pt)$

This motivates the equivariant version of the Hikita conjecture which was proposed by Nakajima:

**Equivariant Hikita Conjecture.** If $X$ and $X^!$ are dual, then

$$B_{\bar{h}} \simeq H^*_{T^! \times \mathbb{C}^\times}(T^*\mathbb{P}^2)$$

as graded algebras over $H^*_{T^! \times \mathbb{C}^\times}(pt)$.