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Category $\mathcal{O}$, symplectic duality, and the Hikita conjecture: Lecture 3
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Recall $\tilde{X} \to X$ conical symplectic resolution, and algebra $A_\hbar$, the Rees algebra of the universal quantization of $\mathbb{C}[X]$. As a reminder, this is a noncommutative filtered algebra whose associated graded in $\mathbb{C}[X]$.

Example 1. $X = \mathbb{C}^2/(\mathbb{Z}/3\mathbb{Z})$ and the resolution

picture

Here

$$A_\hbar = \mathbb{C}[\hbar]\langle a_1, a_2, a_3, b, c\rangle / \langle [a_i, a_j] = 0, [a_i, b] = -\hbar b, [a_i, c] = \hbar c, bc = (a_1 + \hbar)(a_2 + \hbar)(a_3 + \hbar), cb = a_1 a_2 a_3 \rangle$$

This is an algebra over its centre $\mathbb{C}[\hbar, a_1 - a_2, a_2 - a_3] = \mathbb{C}[\hbar] \otimes \mathbb{C}[H^2(\tilde{X})]$. Taking the weight zero space with respect to teh torus action, we get

$$B_\hbar = (A_\hbar)_0 / \langle fg \mid \mathrm{wt}(f) > 0 \ \mathrm{wt}(g) = -\mathrm{wt}(f) \rangle .$$

In our example,

$$\mathbb{C}[\hbar, a_1, a_2, a_3]/\langle bc \rangle = \mathbb{C}[\hbar, a_1, a_2, a_3]/\langle (a_1 + \hbar)(a_2 + \hbar)(a_3 + \hbar) \rangle \simeq H^*_{TX, \mathbb{C}^\times}(T^*\mathbb{P}^2)$$

Equivaraint Hikita is that $B_\hbar \simeq H^*_{T^! \times \mathbb{C}^\times}(\tilde{X}^!)$.

$$S = \mathbb{C}\left\{ q^\lambda \,\middle|\, \lambda \in H_2(\tilde{X}^!; \mathbb{Z}) \right\}.$$

Example 2. $\tilde{X}^! = T^*\mathbb{P}^2$, $H_2(\tilde{X}^!) = \mathbb{Z} \subset \mathbb{N}$ as the effective classes. If $S = \mathbb{C}[q]$, then as a vector space

$$QH^*_{T^! \times \mathbb{C}^\times} = H^*_{T^! \times \mathbb{C}^\times} \otimes \hat{S}$$

with a funny product such if $q = 0$, get usual cup product. The completion means we allow power series.

Theorem 1 (Braverman-Maulik-Okounkov)**.** *"In many cases" there exists a finite set $\Delta_+ \subset H_2(\tilde{X}^!; \mathbb{Z})_{eff}$ and operators for all $\alpha \in \Delta_+$*

$$L_\alpha \colon H^2_{T^! \times \mathbb{C}^\times}(\tilde{X}^!)$$

such that $\forall u \in H^2 T^! \times \mathbb{C}^\times(\tilde{X}^!)$ and for all $vin H^*_{T^! \times \mathbb{C}^\times}(\tilde{X}^!)$, we have

$$u * v = u \cdot v + \sum_{\alpha \in \Delta_+} \langle u, \alpha \rangle \frac{\hbar q^\alpha}{1 - q^\alpha} L_\alpha(v),$$

where we project u to $H^2(\tilde{X}^!)$ and then pair with α. (The power series that appear will all be rational functions).

Example 3. $\tilde{X}^! = T^*\mathbb{P}^2$ and $H^2_{T^! \times \mathbb{C}^\times}(\tilde{X}^!) = \mathbb{C}\{a_1, a_2, a_3, \hbar \mid \}$. $\Delta_+ = \{\alpha\}$ with $\langle a_i, \alpha \rangle = -1$ for all i. So our formula says that

$$a_i * a_i v - \frac{\hbar q}{1 - q} L(V).$$

$L(v) = 0$ for all $v \in H^0$ or H^2. But

$$L(a_1 a_2) = -a_1 a_2 a_3 / \hbar$$

(recall that $(a_1+\hbar)(a_2+\hbar)(a_3+\hbar)$ so the above is divisible by $\hbar$.

Therefore

$$L((a_1+\hbar)(a_3+\hbar)) = L(a_1a_2) + \hbar L(a_2+a_3) + \hbar L(1).$$

Because L kills degree two and zero things, the RHS is $L(a_2a_3)$. The quantum product of the three terms is not zero:

$$(a_1+\hbar)*(a_2+\hbar)*(a_3+\hbar) = (a_1+\hbar)*(a_2+\hbar)(a_3+\hbar) = (a_1+\hbar)(a_2+\hbar)(a_3+\hbar) + \frac{q}{1-q}a_1a_2a_3 = \frac{q}{1-q}a_1a_2a_3.$$

Similarly,

$$a_1a_2a_3 = a_1 * a_2a_3 = a_1a_2a_3 + \frac{q}{1-q}a_1a_2a_3 = \frac{1}{1-q}a_1a_2a_3.$$

Therefore

$$(a_1+\hbar)*(a_2+\hbar)*(a_3+\hbar) = q(a_1*a_2*a_3)$$

and this is the only relation in

$$QH^*_{T^!\times\mathbb{C}^\times}(T^*\mathbb{P}^2) = \mathbb{C}[\hbar, a_1, a_2, a_3, q]/\langle (a_1+\hbar)*(a_2+\hbar)*(a_3+\hbar) = q(a_1*a_2*a_3)\rangle.$$

Recall that $T \subset \mathrm{Aut}(X)$ is a maximal torus, and the basic Hikita conjecture in degree two says that $\mathrm{Lie}(T) \simeq H^2(\tilde{X}^!)$. In particular,

$$\mathrm{Hom}(T, \mathbb{C}^\times) \simeq H_2(\tilde{X}^!; \mathbb{Z}) \supset \Delta_+$$

and the Δ_+ can be thought of as characters of T.

Definition 1. $M := (A_\hbar)_0 \otimes S/S \cdot \left\{ fg - q^\lambda gf \,\middle|\, wt(f) = \lambda \in \mathbb{N}\Delta_+,\ wt(g) = -\lambda \right\}$.

In our favourite example, we have $wt(b) = 1$ and $wt(c) = -1$, so we kill $bc - qcb$. That is,

$$bc - qcb = (a_1+\hbar)*(a_2+\hbar)*(a_3+\hbar) - qa_1a_2a_3.$$

The ring M is interesting in its own right: setting $q = 0$, $M \rightsquigarrow B_\hbar$. Setting $q = 1$, $M \rightsquigarrow HH_0(A_\hbar) = A_\hbar/commutators$, the Hochield homology.

spelling

What is interesting about HH is that if V is finite-dimensional representation of A, we get $A_\hbar$ acting on $V_\hbar = Rees(V)$. We can define a map

$$A_\hbar \to \mathbb{C}[\hbar]$$

sending $f \mapsto \mathrm{tr}(f \text{ acting on } V_\hbar)$. This map factors through HH, and HH can be viewed as being for a universal source of traces.

If $A \circlearrowright V = \bigoplus_{\mu\in\mathrm{Hom}(T,\mathbb{C}^\times)} V_\mu$ with finite- dimensional weight spaces, we get $A_\hbar \circlearrowright V_\hbar$. We get trace maps

$$(A_\hbar)_0 \to \mathbb{C}[\hbar]\otimes\mathbb{C}[q^\mu]$$

a graded version of trace. It sends

$$f \mapsto \sum \mathrm{tr}(f \circlearrowright V_\hbar)_\mu) q^\mu.$$

M can be thought of as a universal source for such traces.

Problem: M is not a ring; what we quotiented by is not an ideal. We have

$$b(a_c) - q(a_c)b = bc(a_1+\hbar) - qa_1cb = (a_1+\hbar)^2(a_2+\hbar)(a_3+\hbar) - qa_1^2a_2a_3.$$

The trick is to change the multiplication.

Definition 2. Let

$$R = \mathbb{C}[\hbar]\left\langle a_1, a_2, a_3, q \,|\, [q, a_i] = q\hbar,\ [a_i, a_j] = 0 \right\rangle,$$

almost polynomial ring, but $qa_i = (a_i + \hbar)q$.

Proposition 1. *M is an R-module, and, in our example, $M \simeq R/R\langle(a_1+\hbar)(a_2+\hbar)(a_3+\hbar) = qa_1a_2a_3\rangle$.*

This makes M nicer, but it is still hard to compare non-commutative R to commutative QH. We will fix this by replacing QH with quantum connection.

Always have a short exact sequence

$$0 \longrightarrow \mathbb{C}\hbar \oplus H_2(\tilde{X}) \longrightarrow (A_\hbar)_0^2 \longrightarrow \mathbb{C}[X]_0^2 \simeq \mathrm{Lie}(T) \longrightarrow 0$$

$$u \longmapsto \bar{u}$$

General presentation of R is then

$$R = S \otimes \mathrm{Sym}(A_\hbar)_0^2$$

with multiplication

$$u \cdot q^\alpha = q^\lambda(u + \langle \lambda, \bar{u}\rangle \,\hbar$$

And $R \circlearrowleft S \otimes (A_\hbar)_0$ with M a quoatient as an R-module.

Dually, $(A_\hbar)_0^2 \simeq H^2_{T^! \times \mathbb{C}^\times}(\tilde{X}^!)$ (equivariant Hikita) and

$$R \circlearrowleft QH^*_{T^! \times \mathbb{C}^\times}(\tilde{X}^!) = \hat{S} \otimes H^*_{T^! \times \mathbb{C}^\times}(\tilde{X}^!$$

by

$$u \cdot (q^\lambda \otimes v) = \hbar \,\langle \lambda, \bar{u}\rangle\, q^\lambda \otimes v + q^\lambda(u * v)$$

this is called the *quantum D-module*

Conjecture 1 (Kamnitzer-McBreen-P)**.** *Quantum Hikita Conjecture: $\hat{H} := \hat{S} \otimes_S M$ is isomorphic to $QH^*_{T^! \times \mathbb{C}^\times}(\tilde{X}^!)$ as a module of $\hat{R} = \hat{S} \otimes_S R$.*

If set $q = 1$ (delicate), M becomes HH.

This is proved for hypertoric varieties and Springer resolution.