Summer School on Geometric Representation Theory
Institute of Science and Technology Austria 2018
Category $\mathcal{O}$, symplectic duality, and the Hikita conjecture

Lecture 1: Category $\mathcal{O}$

Exercises

Let $X := \mathbb{C}^2/(\mathbb{Z}/3\mathbb{Z})$, where a generator $\xi \in \mathbb{Z}/3\mathbb{Z}$ acts on $(x, y) \in \mathbb{C}^2$ by $\xi \cdot (x, y) = (\xi x, \xi^{-1} y)$. Then

$$\mathbb{C}[X] = \mathbb{C}[xy, x^3, y^3] = \mathbb{C}[a, b, c]/\langle a^3 - bc\rangle,$$

with $\deg(a) = 2$ and $\deg(b) = \deg(c) = 3$. This is the Kleinian singularity of type A_2, and it has a unique symplectic resolution $\tilde{X}$ whose fiber over the origin is a pair of projective lines.

1. Compute the Poisson bracket on $\mathbb{C}[X]$.

Hint #1: The Poisson bracket is always a derivation, meaning that $\{f, gh\} = \{f, g\}h + g\{f, h\}$. For this reason, it is enough to compute the brackets of the three generators.

Hint #2: If we regard $\mathbb{C}[X]$ as a subalgebra of $\mathbb{C}[x, y]$, the Poisson bracket $\{f, g\}$ is equal to one third of the coefficient of $dx \wedge dy$ in $df \wedge dg$. (The one third is just there for convenience; we can always introduce a scalar simply by normalizing our generators.)

2. Consider the filtered algebra

$$A := \mathbb{C}\langle a, b, c\rangle \Big/ \Big\langle [a, b] = -b, \;\; [a, c] = c, \;\; bc = (a + 1)(a + 2)(a + 3), \;\; cb = a(a + 1)(a + 2)\Big\rangle,$$

where the i^{th} filtered piece consists of the classes that can be expressed as a (non-commutative) polynomial of degree $\leq i$ in the three generators (where $\deg(a) = 2$ and $\deg(b) = \deg(c) = 3$). Note that we have $[A^i, A^j] \subset A^{i+j-2}$ for all i and j. Show that the associated graded algebra $\operatorname{gr} A$ is isomorphic to $\mathbb{C}[X]$ as a Poisson algebra. In other words, A is a quantization of X.

3. Equip A with an extra $\mathbb{Z}$-grading by putting $\mathrm{wt}(a) = 0$, $\mathrm{wt}(b) = -1$, and $\mathrm{wt}(c) = 1$. This corresponds to the "extra" symplectic action of $\mathbb{C}^\times$ on X and $\tilde{X}$. Let $A_+ \subset A$ be the subalgebra generated by a and c, or equivalently the sum of the non-negative weight spaces. Recall that category $\mathcal{O}$ is defined to be the category of finitely generated A-modules on which A_+ acts locally finitely.

We know that the rank of $K(\mathcal{O})$ is supposed to be equal to the number of irreducible components of $\tilde{X}_+$, which in this case is equal to three (the two projective lines and one affine line).

Let

$$L_1 := A/A \cdot \{a + 1, b, c\}, \qquad L_2 := A/A \cdot \{a + 2, b, c\}, \qquad \text{and} \qquad L_3 := A/A \cdot \{a + 3, c\}.$$

Show that L_1, L_2, and L_3 are each non-zero simple objects of category $\mathcal{O}$. One can show that the localizations of L_1 and L_2 are supported on the two projective lines, and the localization of L_3 is supported on the affine line.