Summer School on Geometric Representation Theory
Institute of Science and Technology Austria 2018
Category $\mathcal{O}$, symplectic duality, and the Hikita conjecture

Lecture 2: The Hikita conjecture

Exercises

1. In the second lecture, we looked at the algebra

$$A_\hbar := \mathbb{C}\langle a_1, a_2, a_3, b, c, \hbar\rangle \Big/ \Big\langle [a,b] = -b\hbar,\ \ [a,c] = c\hbar,\ \ bc = (a_1+\hbar)(a_2+\hbar)(a_3+\hbar),\ \ cb = a_1a_2a_3 \Big\rangle,$$

which is the Rees algebra of the universal quantization of the Kleinian singularity of type A_2. If we set $\hbar = 1$ and $a_3 - a_2 = a_2 - a_1 = 1$, we obtain the algebra A from the first problem set (with $a = a_1$), which is a single quantization of the same space. Applying the construction in the second lecture, we obtain the algebra

$$B_\hbar := \mathbb{C}[a_1, a_2, a_3, \hbar] \Big/ \langle bc\rangle = \mathbb{C}[a_1, a_2, a_3, \hbar] \Big/ \Big\langle (a_1+\hbar)(a_2+\hbar)(a_3+\hbar) \Big\rangle$$

and its specialization

$$B = \mathbb{C}[a, \hbar] \Big/ \langle bc\rangle = \mathbb{C}[a] \Big/ \Big\langle (a+1)(a+2)(a+3) \Big\rangle.$$

Consider the three simple B-modules

$$N_1 := \mathbb{C}[a]/\langle a+1\rangle, \qquad N_2 := \mathbb{C}[a]/\langle a+2\rangle, \qquad \text{and} \qquad N_3 := \mathbb{C}[a]/\langle a+3\rangle.$$

Let $A_- \subset A$ be the subalgebra generated by a and c, and consider the A-modules

$$V_i := A \otimes_{A_-} N_i,$$

where A_- acts on N_i by letting c act by zero. These are the "Verma modules" in category $\mathcal{O}$. Show that each V_i has a unique simple quotient, and that quotient is the module L_i that appeared on the first problem set.[1]

2. In the second lecture, we checked the Hikita conjecture with X equal to the Kleinian singularity of type A_2 and $\tilde{X}^!$ equal to the cotangent bundle of $\mathbb{P}^2$. Now let's check it with the roles reversed. That is, let $\tilde{X}$ be the cotangent bundle of $\mathbb{P}^2$, which means that X is the space of 3×3 nilpotent matrices of rank at most 1. Let $z_{ij} \in \mathbb{C}[X]^2$ be the function that picks out the (i,j) entry of the matrix. The fact that the matrix has rank at most 1 means that $z_{ij}z_{kl} = z_{il}z_{kj}$ for all i, j, k, l, and the fact that it is nilpotent means that in addition we have $z_{11} + z_{22} + z_{33} = 0$.

The torus acting on X is the maximal torus of $\mathrm{PGL}_3(\mathbb{C})$, and the function z_{ij} has weight $e_i - e_j$ with respect to this torus action. Recall that $\mathbb{C}[X^T]$ is the quotient of $\mathbb{C}[X]$ by the ideal generated by all functions of nonzero weight, and it's enough to kill z_{ij} for $i \neq j$. Show that $\mathbb{C}[X^T]$ is isomorphic as a graded algebra to the cohomology ring of the symplectic resolution of the Kleinian singularity of type A_2.

[1] In the first problem set, we called the subalgebra generated by a and c A_+ rather than A_-, which means that we were using the opposite $\mathbb{Z}$-grading on A and the corresponding opposite definition of category $\mathcal{O}$. These two differences cancel out, so the category $\mathcal{O}$ that appears in both problem sets is the same!