Category $\mathcal{O}$, symplectic duality and the Hikita conjecture

Lecture 1

Let G be an algebraic group, simple over $\mathbb{C}$ (e.g. $G = SL_n(\mathbb{C})$), $B \subseteq G$ a Borel subgroup, $Y = G/B$ the flag variety. Let

$$\tilde{X} = T^*Y = \{(gB, a) \in Y \times \mathcal{N} \mid g^{-1}ag \in \mathfrak{b}\} \longrightarrow X = \mathcal{N},$$ the Springer resolution.

Let $Z := \tilde{X} \times_X \tilde{X}$ be the Steinberg variety. This admits a decomposition as $Z = \bigcup\{\text{conormal bundles to } G\text{-orbits on } Y \times Y\}$.

Example: For $G = SL_2(\mathbb{C})$, $\tilde{X} = T^*\mathbb{P}^1 \longrightarrow X = \mathbb{C}^2/\{\pm 1\}$. $Z = T^*\mathbb{P}^1 \cup_{\mathbb{P}^1} (\mathbb{P}^1 \times \mathbb{P}^1)$, where $\mathbb{P}^1$ embeds into $T^*\mathbb{P}^1$ as the zero section.

Note that all irreducible components of Z have the same dimension $d = \dim X$.
Let $\tilde{X}_+ = \bigcup\{\text{conormal bundles to } B\text{-orbits on } Y\}$. If $Y = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, then $\tilde{X}_+ = \mathbb{C} \cup T^*_\infty \mathbb{CP}^1$.

Consider the top Borel-Moore homology of Z: $H_{2d}^{BM}(Z) = \mathbb{C}^{|\text{components}|}$.
A priori, this is a $\mathbb{C}$-vector space, but it has an algebraic structure given by:

$$\begin{array}{ccccc} & & \tilde{X} \times_X \tilde{X} \times_X \tilde{X} & & \\ & \swarrow^{p_{12}} & \downarrow^{p_{13}} & \searrow^{p_{23}} & \\ Z & & Z & & Z \end{array}$$

$$\alpha * \beta = (p_{13})_* (p_{12}^* \alpha - p_{23}^* \beta).$$

There is an action $H_{2d}^{BM}(Z) \curvearrowright H_d^{BM}(\tilde{X}_+)$ by a similar construction.

Theorem: $H_{2d}^{BM}(Z) \simeq \mathbb{C}[W]$ (as algebras)
$H_d^{BM}(\tilde{X}^+) \simeq \mathbb{C}[W]$ (as $\mathbb{C}[W]$-modules)

Given a group action $G \curvearrowright Y$, there is an induced map $U(\mathfrak{g}) \xrightarrow{\varphi} \Gamma(Y, \mathcal{D}_Y)$, which is surjective. Set $U(\mathfrak{g})_0 := U(\mathfrak{g})/\ker \varphi \simeq \Gamma(Y, \mathcal{D}_Y)$.

Theorem: The map $(U(\mathfrak{g})_0\text{-mod}) \xrightarrow{\mathrm{Loc}} \mathcal{D}_Y\text{-mod}$, $N \longmapsto \mathcal{D}_Y \otimes_{U(\mathfrak{g})_0} N$ is an equivalence of categories.

We have maps $U(\mathfrak{g})_0\text{-mod} \xrightarrow{\mathrm{Loc}} \mathcal{D}_Y\text{-mod} \xrightarrow[\text{support}]{\text{microlocal}} \{\text{cycles on } \tilde{X} = T^*Y\}$.

$$\mathcal{O}_0 \longrightarrow \{\text{cycles on } \tilde{X}_+\}.$$

Here, $\mathcal{O}_0$ denotes the category of finitely generated $U(\mathfrak{g})_0$-modules that are locally finite for $U(\mathfrak{b}) \subseteq U(\mathfrak{g})$. Furthermore:

$$K(\mathcal{O}_0) \otimes \mathbb{C} \xrightarrow{\sim} H_d^{BM}(\tilde{X}_+)$$

Definition: The Harish-Chandra modules $HC_0 :=$ finitely generated $U(\mathfrak{g})_0$-bimodules that are locally finite for the adjoint action.

This yields maps
$$\begin{array}{ccccc} U(\mathfrak{g})_0\text{-mod} & \xrightarrow{\text{Loc}} & D_Y \boxtimes D_Y^{op}\text{-mod} & \rightsquigarrow & \{\text{cycles on } \tilde{X}\times\tilde{X}\} \\ \cup\!| & & & & \cup\!| \\ HC_0 & & \longrightarrow & & \{\text{cycles on } Z\} \end{array}$$

Theorem: (i) HC_0 is a tensor category acting on $\mathcal{O}_0$.

(ii) Support intertwines $\otimes^L$ with $*$.

(iii) There are bimodules $\{H_w \mid w\in W\}$ s.t.

(a) $\Theta_w : D^b(\mathcal{O}_0) \xrightarrow{H_w\otimes^L -} D^b(\mathcal{O}_0)$ is an equivalence of categories.

(b) $\Theta_w \cdot \Theta_{w'} = \Theta_{ww'}$ for $l(w)+l(w') = l(ww')$

So $B_W \circlearrowright D^b(\mathcal{O}_0)$, categorifying $W \circlearrowright K(\mathcal{O}_0)$

Definition: A conical symplectic resolution is a resolution of singularities $\tilde{X}\to X$, with an action of $\mathbb{C}^\times = S$, and a symplectic form $\omega\in\Omega^2(\tilde{X})$, s.t:

(i) X is a normal affine cone: $\mathbb{C}[X] = \bigoplus_{n\geq 0}\mathbb{C}[X]^n$, $\mathbb{C}[X]^0 = \mathbb{C}$, $\mathbb{C}[X]^1 = 0$.

(ii) ω has weight 2: $s.\omega = s^2\omega \ \forall s\in S$

Examples: (i) $\tilde{X} = T^* G/B$: S acts on the fibres with weight 2.

(ii) $\tilde{X} = \mathrm{Hilb}^n\mathbb{C}^2$, $X = \mathrm{Sym}^n\mathbb{C}^2$.

(iii) Quiver varieties

(iv) Slices in the affine Grassmannian

(v) Hypertoric varieties: from combinatorics

(vi) Higgs/Coulomb moduli spaces

We also need an extra $\mathbb{C}^*$-action on $\tilde{X}\to X$, commuting with S and preserving ω: $[|X^{\mathbb{C}^\times}|<\infty]$

$\mathbb{C}^\times \hookrightarrow G \circlearrowright G/B \rightsquigarrow \mathbb{C}^\times \circlearrowright T^* G/B$.

Given $Z = \tilde{X}\times_X\tilde{X}$, $\tilde{X}_+ = \{p\in\tilde{X} \mid \lim_{t\to 0} t.p \text{ exists}\}$, $t\in\mathbb{C}^\times$ the 'extra action'. Then $H^{BM}_{2d}(Z) \circlearrowright H^{BM}_d(\tilde{X}_+)$ as before.

Definition: A quantisation of $\tilde{X}$ is a sheaf $\mathcal{A}$ of filtered algebras on $\tilde{X}$ s.t. $[\mathcal{A}^i, \mathcal{A}^j] \subseteq \mathcal{A}^{i+j-2}$, and an isomorphism $\mathrm{gr}\,\mathcal{A} \simeq \mathcal{O}_{\tilde{X}}$ of graded Poisson algebras.

Let $A = \Gamma(\tilde{X},\mathcal{A})$: A is a filtered algebra with $\mathrm{gr}\,A \simeq \mathbb{C}[X]\ (=\mathbb{C}[\tilde{X}])$. If $\tilde{X} = T^* G/B$, $\mathcal{A} = \pi^{-1}\mathcal{D}_{G/B}$, $A = \Gamma(G/B, \mathcal{D}_{G/B}) \simeq U(\mathfrak{g})_0$.

There is an action (unique) $\mathbb{C}^\times \circlearrowright A$ s.t. $\mathrm{gr}\, A \simeq \mathbb{C}[X]$ is compatible with the 'extra grading'.

<u>Theorem</u>: The functor $A\text{-mod} \longrightarrow \mathcal{A}\text{-mod}$, $N \longmapsto \mathcal{A} \otimes_A N$, is an equivalence for 'most' quantisations.

We have maps

$$\begin{tikzcd} A\text{-mod} \arrow[r] & \mathcal{A}\text{-mod} \arrow[r, squiggly] & \{\text{cycles on } \tilde{X}\} \\ \mathcal{O} \arrow[u, phantom, "\cup"] \arrow[rr, squiggly] & & \{\text{cycles on } \tilde{X}_+\} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

where $\mathcal{O}$ is the category of A-modules that are locally finite for the action of $A_+ \subseteq A$ (sum of non-negative weight spaces for extra action).

<u>Proposition</u>: $K(\mathcal{O}) \otimes \mathbb{C} \simeq H_d^{BM}(\tilde{X}_+)$.

$$\begin{tikzcd} A\text{-bimod} \arrow[r, "\mathrm{Loc}"] & \mathcal{A}\text{-bimod} \arrow[r, squiggly] & \{\text{cycles on } \tilde{X} \times \tilde{X}\} \\ HC \arrow[u, phantom, "\cup"] \arrow[rr, squiggly] & & \{\text{cycles on } Z\} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

where $HC = \mathrm{gr}$ supported on X_Δ.

<u>Theorem</u>: (i) HC is a tensor category acting on $\mathcal{O}$
(ii) Support intertwines $\otimes$ with $*$.
(iii) There are nice HC-bimodules that fit together into a generalised braid group action on $D^b(\mathcal{O})$

<u>Lecture 2</u>

Let $\mathrm{Aut}(X)$ be the conical symplectic automorphisms of $\tilde{X}$, $T \subseteq \mathrm{Aut}(X)$ a maximal torus. Suppose $|\tilde{X}^T| < \infty$ and choose a generic cocharacter $\mathbb{C}^\times \hookrightarrow T$ (the 'extra action').

<u>Examples</u>: (i) Let $X = \mathbb{C}^2/(\mathbb{Z}/3\mathbb{Z}) = \mathrm{Spec}\, \mathbb{C}[x,y]^{\mathbb{Z}/3\mathbb{Z}} = \mathrm{Spec}\, \mathbb{C}[xy, x^3, y^3]$

$= \mathrm{Spec}\left(\frac{\mathbb{C}[a,b,c]}{\langle a^3 - bc \rangle}\right)$. This has a natural symplectic action via $\zeta \cdot x = \zeta x$, $\zeta \cdot y = \zeta y^{-1}$.

S scales $\mathbb{C}^2$, so $\deg(a) = 2$, $\deg(b) = \deg(c) = 3$

$$\begin{array}{c} \tilde{X} \\ \downarrow \\ X \end{array} \quad T = \mathrm{Aut}(X) = \mathbb{C}^\times, \text{ with: } t \cdot a = a,\ t \cdot b = b,\ t \cdot c = tc^{-1}.$$

Institute of Science and Technology Austria | Am Campus 1 | 3400 Klosterneuburg | Austria
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(ii) Let $X^! := \{3\times 3$ nilpotent matrices of rank $\leq 1\}$. The resolution:

$$\begin{tikzcd} & \widetilde{X^!} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X^! & & \mathbb{P}^2 \end{tikzcd}$$

$\widetilde{X^!} = \{(M, L) \mid M \in X^!, L \text{ line } \in \operatorname{im}(M)\}$.

Then $\widetilde{X^!} \simeq T^*\mathbb{P}^2$, with π_1 the resolution of singularities collapsing the zero section to a point. S scales fibres with weight -2.

$\operatorname{Aut}(X^!) = PGL_3(\mathbb{C})$, $T^! = \dfrac{(\mathbb{C}^\times)^3}{\mathbb{C}^\times_\Delta}$ is 2-dimensional. $(\widetilde{X^!})^{T^!}$ coordinate points on $\mathbb{P}^2$. X, $X^!$ are <u>dual</u> in the sense that $\mathcal{O}_X$ is Koszul dual to $\mathcal{O}_{X^!}$.

Other examples of dual pairs include:

(i) $T^*(G/B)$ is dual to $T^*\left(\frac{G^\vee}{B^\vee}\right)$ $(\simeq T^*(G/B))$

(ii) $\operatorname{Hilb}^n \dfrac{\mathbb{C}^2}{\mathbb{C}^2}$ is self-dual.

(iii) Quiver varieties are dual to slices in the affine Grassmannian Gr_G.

(iv) Hypertoric varieties are dual to other hypertoric varieties.

<u>Q</u>: What is the coordinate ring of X^T? Note that $|\widetilde{X}^T| < \infty =: N$, so the coordinate ring of $|\widetilde{X}^T|$ is always isomorphic to $\mathbb{C}^N$.

Suppose G acts on $X =: \operatorname{Spec} R$, so $p \in X^G \iff f(p) = f(\sigma p)\ \forall f \in R, \forall \sigma \in G$, and is equivalently ~~the~~ same as $f(p) = (\sigma\cdot f)(p)$. We set:

$$X^G := \frac{\operatorname{Spec} R}{\langle \sigma f - f \mid \sigma, f\rangle}.$$

If $G = \mathbb{C}^\times$, then $\langle \sigma\cdot f - f\rangle = \langle$ all homogeneous functions of weight $\neq 0\rangle$. Then

$$X^{\mathbb{C}^*} = \frac{\operatorname{Spec} \mathbb{C}[X]_0}{\langle fg \mid wt(f) = -wt(g) \neq 0\rangle}.$$

<u>Example</u>: Let $\mathbb{C}[X] = \dfrac{\mathbb{C}[a,b,c]}{\langle a^3 - bc\rangle}$, $wt(a) = 0$, $wt(b) = 1$, $wt(c) = -1$. Then

$$\mathbb{C}[X^T] = \frac{\mathbb{C}[a]}{\langle bc\rangle} = \frac{\mathbb{C}[a]}{\langle a^3\rangle} = H^*(\widetilde{X^!})$$

<u>Conjecture</u>: [Hikita] If X, $X^!$ are dual, $\mathbb{C}[X^T] \simeq H^*(\widetilde{X^!})$ as graded $\mathbb{C}$-algebras.

This has been proved in certain special cases:

(i) the Springer resolution

(ii) $\operatorname{Hilb}^n \dfrac{\mathbb{C}^2}{\mathbb{C}^2}$

(iii) finite type A quiver varieties / Gr_G slices

(iv) hypertoric varieties

In degree 2, $H^2(\tilde{X}^{\cdot}) \simeq \mathbb{C}[X]^2_0$ (since $\mathbb{C}[X]^1_0 = 0$ by assumption)
$\simeq \mathrm{Lie}(T)$ (via the moment map).

Theorem: There is a 1-1 correspondence $\{\text{quantisations of } X\} \xleftrightarrow{\text{period}} H^2(\tilde{X})$.

Example: Let $\mathbb{C}[X] = \dfrac{\mathbb{C}[a, b, c]}{\langle a^3 - bc\rangle}$. Then $\{a, b\} = -b$, $\{a, c\} = c$, $\{b, c\} = 3a^2$.

Then $\mathbb{C}[X] = \dfrac{\mathbb{C}[a_1, a_2, a_3, b, c]}{\langle a_1 a_2 a_3 bc, a_1 - a_2, a_1 - a_3\rangle}$

Let $A = \dfrac{\mathbb{C}\langle a_1, a_2, a_3, b, c\rangle}{\left\langle \begin{array}{l}[a_i, a_j] = 0, \ [a_i, b] = -b, \ [a_i, c] = c \\ bc = (a_1+1)(a_2+1)(a_3+1), \ cb = a_1 a_2 a_3\end{array}\right\rangle}$.

The centre $Z(A) = \mathbb{C}[a_1 - a_2, a_2 - a_3] \simeq \mathbb{C}[H^2(\tilde{X})]$.
If we set x, y equal to specific complex numbers, we get a quantisation of $\tilde{X}$.

Definition: The Rees operator of A is $A_\hbar = \sum_{i \in I} \hbar^i A^i \subseteq \mathbb{C}[\hbar] \otimes A$.
This is algebraic over $\mathbb{C}[\hbar]$: if $\hbar = 1$ $A_1 = A$, and $A_0 = \mathrm{gr}\, A$.
This is algebraic over $\mathbb{C}[H^2(\tilde{X})] \otimes \mathbb{C}[\hbar]$.

Set $B_\hbar := \dfrac{(A_\hbar)_0}{\langle fg \mid \mathrm{wt}(f) = -\mathrm{wt}(g) > 0\rangle}$. This is an algebra over $\mathbb{C}[H^2(\tilde{X})] \otimes \mathbb{C}[\hbar]$.
Note we have natural surjections:

$(A_\hbar)_- \twoheadrightarrow (A_\hbar)_0 \twoheadrightarrow (B_\hbar)_0$. Given a $B_\hbar$-module N,
$V = A_\hbar \otimes_{(A_\hbar)_-} N$ is a 'Verma module'.

Example: As before, $A_\hbar = \dfrac{\mathbb{C}\langle a_1, a_2, a_3, b, c\rangle[\hbar]}{\left\langle \begin{array}{l}[a_i, a_j] = 0, \ [a_i, b] = -b\hbar, \ [a_i, c] = c\hbar, \\ bc = (a_1+\hbar)(a_2+\hbar)(a_3+\hbar), \ cb = a_1 a_2 a_3\end{array}\right\rangle}$:

$$B_\hbar = \frac{\mathbb{C}[\hbar, a_1, a_2, a_3]}{\langle bc\rangle} = \frac{\mathbb{C}[\hbar, a_1, a_2, a_3]}{\langle (a_1+\hbar)(a_2+\hbar)(a_3+\hbar)\rangle} \simeq H_{T \times \mathbb{C}^\times}(T^*\mathbb{P}^2).$$

Institute of Science and Technology Austria | Am Campus 1 | 3400 Klosterneuburg | Austria
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This is an algebra over $\mathbb{C}[H^2(\tilde{X})] \otimes \mathbb{C}[\hbar] \simeq \mathbb{C}[\mathrm{Lie}\, T] \otimes \mathbb{C}[\hbar]$
$\simeq H_{T^! \times \mathbb{C}^\times}(pt)$.

<u>Conjecture</u>: $B_\hbar \simeq H_{T^! \times \mathbb{C}^\times}(\widehat{X}^!)$ as graded algebras over $H_{T^! \times \mathbb{C}^\times}(pt)$.
[Equivariant Hikita]

<u>Lecture 3</u>

Let $\tilde{X} \to X$ be a conical symplectic resolution, $A_\hbar$ the Rees algebra of the universal quantisation of $\mathbb{C}[X]$; this is an algebra over $\mathbb{C}[\hbar, a_1 - a_2, a_2 - a_3]$ $= \mathbb{C}[\hbar] \otimes \mathbb{C}[H^2(\tilde{X})]$. Set $B_\hbar = (A_\hbar)_0 / \langle fg \mid wt(f) > 0, wt(g) = -wt(f) \rangle$.

Let $S := \mathbb{C}[q^\lambda \mid \lambda \in H_2(\widehat{X}^!, \mathbb{Z})$ effective$]$. If $\widehat{X}^! = T^*\mathbb{P}^2$ as before, $H_2(\widehat{X}^!, \mathbb{Z}) = \mathbb{Z} \supseteq \mathbb{N}$, the effective divisors, and $S = \mathbb{C}[q]$.

The <u>quantum cohomology</u> is $QH^*_{T^! \times \mathbb{C}^\times}(\widehat{X}^!) = H^*_{T^! \times \mathbb{C}^\times}(\widehat{X}^!) \otimes \widehat{S}$, with a product s.t. $q = 0$ reduces to the usual product.

<u>Theorem</u>: In many cases, there exist:
(i) a finite set $\Delta_+ \subseteq H_2(\widehat{X}^!, \mathbb{Z})_{\mathrm{eff}}$
(ii) operators $L_\alpha : H^*_{T^! \times \mathbb{C}^\times}(\widehat{X}^!)$ for $\alpha \in \Delta_+$ s.t. for $u \in H^2_{T^! \times \mathbb{C}^\times}(\widehat{X}^!)$, $v \in H^*_{T^! \times \mathbb{C}^\times}(\widehat{X}^!)$:

$$u *_q v = u \cdot v + \sum_{\alpha \in \Delta_+} \langle u, \alpha \rangle \frac{\hbar q^\alpha}{1 - q^\alpha} L_\alpha(v),$$

where $\langle u, \alpha \rangle$ is the projection to $H^2(\widehat{X}^!)$ and then paired with α.

<u>Example</u>: If $\widehat{X}^! = T^*\mathbb{P}^2$, $H^2_{T^! \times \mathbb{C}^\times}(\widehat{X}^!) = \mathbb{C}\{a_1, a_2, a_3, \hbar\}$. $\Delta_+ = \{\alpha\}$ with $\langle a_i, \alpha \rangle = -1 \ \forall i$. Then $a_i * v = a_i v - \frac{\hbar (q)}{1-q} L(v)$, and one computes:
$L(v) = 0 \ \forall v \in H^0 \cup H^2$, $L(a_2 a_3) = \frac{-a_1 a_2 a_3}{\hbar}$.

$L((a_2 + \hbar)(a_3 + \hbar)) = L(a_2 a_3) + \hbar L(a_2 + a_3) + \hbar^2 L(1) = L(a_2 a_3)$, since the other two terms vanish.

Recall $(a_1 + \hbar)(a_2 + \hbar)(a_3 + \hbar) = 0$. One computes:
$(a_1 + \hbar) * (a_2 + \hbar) * (a_3 + \hbar) = (a_1 + \hbar) * (a_2 + \hbar)(a_3 + \hbar)$
$= (a_1 + \hbar)(a_2 + \hbar)(a_3 + \hbar) + \frac{q}{1-q} a_1 a_2 a_3$

and: $a_1 * a_2 * a_3 * = a_1 * a_2 a_3 = a_1 a_2 a_3 + \frac{q}{1-q} a_1 a_2 a_3 = \frac{1}{1-q} a_1 a_2 a_3$

Hence $(a_1+\hbar)*(a_2+\hbar)*(a_3+\hbar) = q a_1 * a_2 * a_3$, and:

$$QH^*_{T^! \times \mathbb{C}^\times}(T^*\mathbb{P}^2) = \frac{\mathbb{C}[\hbar, a_1, a_2, a_3, q]}{\langle (a_1+\hbar)(a_2+\hbar)(a_3+\hbar) - q a_1 a_2 a_3 \rangle}.$$

Recall that $T \subseteq Aut(X)$ is a maximal torus, and the basic Hikita conjecture states that $Lie(T) \simeq H^2(\tilde{X}^!)$, so $Hom(T, \mathbb{C}^\times) \simeq H_2(\tilde{X}^!; \mathbb{Z}) \supseteq \Delta_+$.

Set:

$$M := \frac{(A_\hbar)_0 \otimes S}{S \cdot \{ fg - q^\lambda gf \mid wt(f) = \lambda \in \mathbb{N}\Delta_+, \ wt(g) = -\lambda \}}$$

In the example, $wt(b) = 1$, $wt(c) = -1$, so we kill $bc - qcb = (a_1+\hbar)(a_2+\hbar)(a_3+\hbar) - q a_1 a_2 a_3$. Note, if $q = 0$, $M = B_\hbar$, and if $q = 1$, $M = HH_0(A_\hbar) = \frac{A_\hbar}{\text{commutators}}$, the Hochschild cohomology of $A_\hbar$.

If A acts on a finite-dimensional module V, $A_\hbar$ acts on $V_\hbar = Rees(V)$, via $f \longmapsto tr(f \circlearrowright V_\hbar)$.

$$\begin{tikzcd} A_\hbar \arrow[rr] \arrow[dr] & & \mathbb{C}[\hbar] \\ & HH_0(A_\hbar) \arrow[ur] & \end{tikzcd}$$

More generally, if $V = \bigoplus_{\mu \in Hom(T, \mathbb{C}^\times)} V_\mu$ is a direct sum of finite-dimensional weight spaces, $A_\hbar$ acts on $V_\hbar$ via $(A_\hbar)_0 \longrightarrow \mathbb{C}[\hbar] \otimes \mathbb{C}[q^\mu]$, $f \longmapsto \sum_\mu tr(f \circlearrowright (V_\hbar)_\mu) q^\mu$

M is not a ring: note that $b(a_1 c) - q(a_1 c) b = bc(a_1+\hbar) - q a_1 cb = (a_1+\hbar)^2 (a_2+\hbar)(a_3+\hbar) - q a_1^2 a_2 a_3$. Set:

$$R := \frac{\mathbb{C}[\hbar]\langle a_1, a_2, a_3, q \rangle}{\langle [q, a_i] = q\hbar, \ [a_i, a_j] = 0 \rangle}.$$

Then $q a_i = (a_i + \hbar) q$.

Institute of Science and Technology Austria | Am Campus 1 | 3400 Klosterneuburg | Austria
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<u>Proposition</u>: M is an R-module, and in our example:

$$M \simeq \frac{R}{R\cdot\{(a_1+\hbar)(a_2+\hbar)(a_3+\hbar) - q a_1 a_2 a_3\}}$$

Recall there is a short exact sequence $0 \longrightarrow \mathbb{C}\hbar \oplus H_2(\tilde{X}) \longrightarrow (A_\hbar)^2_0 \longrightarrow \mathbb{C}[X]^2_0 \longrightarrow 0$.

Then set $R = S \otimes \mathrm{Sym}(A_\hbar)^2_0$, $u \cdot q^\alpha = q^\lambda(u + \langle \lambda, \bar{u} \rangle \hbar)$.

R acts on $S \otimes (A_\hbar)_0 \twoheadrightarrow M$, and $(A_\hbar)^2_0 \simeq H^2_{T^! \times \mathbb{C}^\times}(\tilde{X}^!)$ by equivariant Hikita. Then R acts on $QH^*_{T^! \times \mathbb{C}^\times}(\tilde{X}^!) = \hat{S} \otimes H^*_{T^! \times \mathbb{C}^\times}(\tilde{X}^!)$ via.

$$u \cdot (q^\lambda \otimes v) = \hbar \langle \lambda, \bar{u} \rangle q^\lambda \otimes v + q^\lambda \otimes (u * v) \quad \text{[quantum D-module]}$$

<u>Conjecture</u>: $\hat{M} := \hat{S} \otimes_S M$ is isomorphic to $QH^*_{T^! \times \mathbb{C}^\times}(\tilde{X}^!)$ as a module over
[Quantum Hikita] $\hat{R} := \hat{S} \otimes_S R$.