Category $\mathcal{O}$, symplectic duality, and the Hikita Conjecture.

Three lectures at IST Austria, July 2018.

Lecture 1: Category $\mathcal{O}$

I want to talk about symplectic resolutions in general, but I'll start by talking about one particular example, namely the Springer resolution.

G simple alg. group $/\mathbb{C}$ (eg $SL_n(\mathbb{C})$)

$B \subset G$ Borel

$Y = G/B$ (eg $\mathrm{Flag}(\mathbb{C}^n)$)

$$\begin{array}{l} \tilde{X} = T^*Y \cong \{(gB, a) \in Y \times \mathrm{nil}(\mathfrak{g}) \mid g^{-1}ag \in \mathfrak{b}\} \\ \downarrow \\ X = \mathrm{nil}(\mathfrak{g}) \end{array}$$

cotangent bundle is a subbundle of the trivial bundle

$Z = \tilde{X} \times_X \tilde{X} = \bigcup$ conormal bundles to G-orbits in $Y \times Y$

Ex. $G = SL_2(\mathbb{C})$

$$\begin{array}{l} \tilde{X} = T^*\mathbb{P}^1 \\ \downarrow \\ X = \mathbb{C}^2/\pm 1 \end{array} \qquad Z = T^*\mathbb{P}^1 \underset{\mathbb{P}^1}{\cup} \mathbb{P}^1 \times \mathbb{P}^1$$

Fact: All irred. components of Z have dimension $d = \dim X$.

$\tilde{X}_+ = \bigcup$ conormal bundles to B-orbits in Y $\subset \tilde{X}$

Ex: $Y = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$

$\tilde{X}_+ = \mathbb{C} \cup T^*_\infty \mathbb{CP}^1$

Consider the top degree Borel-Moore homology group

$$H_{2d}^{BM}(Z) = \mathbb{C}^{\#\text{ components}}$$

Algebra structure:

$$\begin{tikzcd} & \tilde{X} \times_X \tilde{X} \times_X \tilde{X} \arrow[dl, "p_{12}"'] \arrow[d, "p_{13}"] \arrow[dr, "p_{23}"] & \\ Z & Z & Z \end{tikzcd}$$

$$\alpha * \beta = (p_{13})_* \left(p_{12}^* \alpha \cdot p_{23}^* \beta \right)$$

need to be a little delicate here, doing the intersection with supports inside of $\tilde{X} \times \tilde{X} \times \tilde{X}$.

I also want to think about the top Borel-Moore homology of $\tilde{X}^+$, which is a module over this algebra by a similar construction.

$$H_{2d}^{BM}(Z) \circlearrowright H_d^{BM}(\tilde{X}_+)$$

Thm (Lusztig, Ginzburg, ...): $H_{2d}^{BM}(Z) \cong \mathbb{C}[W]$

$$H_d^{BM}(\tilde{X}_+) \cong \mathbb{C}[W]$$

So the regular representation of the Weyl group can be realized by convolution operators acting on the Borel-Moore homology of $\tilde{X}_+$.

Why should we care? Geometrization $\rightsquigarrow$ Categorification.

$\mathfrak{g} \curvearrowright Y \rightsquigarrow U(\mathfrak{g}) \longrightarrow \mathrm{Diff}(Y) = \Gamma(Y; D_Y)$.

Fact: This map is surjective.
Let $U(\mathfrak{g})_0 = U(\mathfrak{g})/\ker \cong \mathrm{Diff}(Y)$

I want to use this notation to emphasize that it is a quotient of $U(\mathfrak{g})$.

Thm (Beilinson-Bernstein): $U(\mathfrak{g})_0\text{-mod} \xrightarrow{\mathrm{Loc}} D_Y\text{-mod}$

$$N \longmapsto D_Y \underset{U(\mathfrak{g})_0}{\otimes} N$$

is an equivalence of categories.

Thus representations of $\mathfrak{g}$ can be interpreted as sheaves on the flag variety Y. In particular, we have a notion of "support"

$$U(\mathfrak{g})_0\text{-mod} \xrightarrow{\mathrm{Loc}} D_Y\text{-mod} \underset{\text{"microlocal support"}}{\rightsquigarrow} \text{cycles on } \tilde{X} = T^*Y$$

We're interested in cycles on $\tilde{X}_+$, so let's define a subcategory on the left whose microlocal support cycles will land in $\tilde{X}_+$

Let $\mathcal{O}_0$ = f.g. $U(\mathfrak{g})_0$-modules that are locally finite for the action of $U(\mathfrak{b}) \subset U(\mathfrak{g})$.

$$\begin{array}{ccccc} U(\mathfrak{g})_0\text{-mod} & \xrightarrow{\text{Loc}} & D_Y\text{-mod} & \underset{\text{"microlocal support"}}{\rightsquigarrow} & \text{cycles on } \tilde{X} = T^*Y \\ \cup & & & & \cup \\ \text{Fact: } \mathcal{O}_0 & & \rightsquigarrow & & \text{cycles on } \tilde{X}_+ \end{array}$$

$$K(\mathcal{O}_0)\otimes\mathbb{C} \xrightarrow{\approx} H_d^{BM}(\tilde{X}_+)$$

The point is that local finiteness for $U(\mathfrak{b})$ let's you integrate to a B-action, which guarantees that the microlocal support lies in the union of the conormal bundles to the B-orbits.

Next goal: Lift the convolution operators on $H_d^{BM}(\tilde{X}_+)$ to functors acting on $\mathcal{O}_0$. Look at bimodules!

$$U(\mathfrak{g})_0\text{-bimod} \xrightarrow{\text{Loc}} D_Y \boxtimes D_Y^{op}\text{-mod} \rightsquigarrow \text{cycles on } \tilde{X}\times\tilde{X}$$

But we want cycles on Z...

Def: HC_0 = f.g. $U(\mathfrak{g})_0$-bimodules that are locally finite for the adjoint action.

$$\begin{array}{ccccc} U(\mathfrak{g})_0\text{-bimod} & \xrightarrow{\text{Loc}} & D_Y \boxtimes D_Y^{op}\text{-mod} & \rightsquigarrow & \text{cycles on } \tilde{X}\times\tilde{X} \\ \cup & & & & \cup \\ \text{Fact: } HC_0 & & \rightsquigarrow & & \text{cycles on } Z \end{array}$$

The explanation is the same as before: local finiteness implies that we can integrate to a G-action, which means that the microlocal support lies in the union of the conormal bundles of the G-orbits.

Thm: 1 HC_0 is a tensor category acting on $\mathcal{O}_0$.

2 Support intertwines $\overset{L}{\otimes}$ with $*$

3 $\exists$ bimodules $\{H_w \mid w \in W\}$ st

- $\Theta_w : D^b(\mathcal{O}_0) \xrightarrow{H_w \overset{L}{\otimes} -} D^b(\mathcal{O}_0)$ is an equivalence
- $\Theta_w \circ \Theta_{w'} = \Theta_{ww'}$ whenever $\ell(w) + \ell(w') = \ell(ww')$

Thus $B_W \curvearrowright D^b(\mathcal{O}_0)$

categorifying $W \curvearrowright K(\mathcal{O}_0) \otimes \mathbb{C} \cong \mathbb{C}[W]$.

"twisting functors"

For example, a simple transposition acts as an involution on the Grothendieck group, but not on the category. The categorical version is more interesting!

Now I want to generalize all of this to other symplectic resolutions!

Def: A conical symplectic resolution is

- A resolution $\begin{array}{c}\tilde{X} \\ \downarrow \\ X\end{array} \circlearrowleft S = \mathbb{C}^*$
- A symplectic form $\omega \in \Omega^2(\tilde{X})$

Satisfying:

- X normal affine cone: $\mathbb{C}[X] = \bigoplus_{n \geq 0} \mathbb{C}[X]^n$

  $\mathbb{C}[X]^0 = \mathbb{C}$.
- ω has weight 2: $s \cdot \omega = s^2 \omega \quad \forall s \in S$

In the next lecture and on the exercises I'll give some very concrete examples! For now, let me just name a few examples to give you an idea of where such things come from.

Ex: ① $\tilde{X} = T^*(G/B)$

$X = \mathcal{N}(\mathfrak{g})$ $\quad$ S scales the fibers with weight -2.

② $\tilde{X} = \mathrm{Hilb}^n \mathbb{C}^2$ $\quad$ S scales $\mathbb{C}^2$ with weight -1.

$X = \mathrm{Sym}^n \mathbb{C}^2$

③ Quiver varieties
④ "Slices" in Gr_G } Rep Theory

⑤ Hypertoric varieties } Combinatorics

⑥ Higgs/Coulomb moduli spaces of gauge theories } physics

One more piece of data:

"Extra" $\mathbb{C}^\times \curvearrowright \begin{array}{c}\tilde{X}\\ \downarrow \\ X\end{array}$, commuting with S, preserving ω. Assume $|\tilde{X}^{\mathbb{C}^\times}| < \infty$.

Ex ① $\mathbb{C}^\times \hookrightarrow G \curvearrowright G/B$
$\rightsquigarrow \mathbb{C}^\times \curvearrowright T^*(G/B)$

Such an extra $\mathbb{C}^\times$ action doesn't always exist, but I want to assume it does and choose one

Let $Z = \tilde{X} \times_X \tilde{X}$

$$\tilde{X}_+ = \{ p \in \tilde{X} \mid \lim_{\substack{t \to 0 \\ \cap \\ \mathbb{C}^\times}} t \cdot p \text{ exists} \}$$

Ex: $\tilde{X} = T^*\mathbb{P}^1$, $\mathbb{C}^\times \curvearrowright \mathbb{P}^1$, $\tilde{X}_+ =$  as before

Then $H^{BM}_{2d}(Z) \curvearrowright H^{BM}_d(\tilde{X}_+)$ as before.

I again want to define a category that has $H^{BM}_d(\tilde{X}_+)$ as its Grothendieck group. We can't do D-modules on the base anymore, because there is no base! Instead, we think about quantizations.

Def: A quantization of $\tilde{X}$ is a sheaf $\mathcal{A}$ of filtered algebras $\mathcal{A}$ on $\tilde{X}$ with $[\mathcal{A}^i, \mathcal{A}^j] \subset \mathcal{A}^{i+j-2}$ plus an isom $\mathrm{gr}\,\mathcal{A} \cong \mathrm{Fun}_X$ of graded Poisson algebras

What does this mean? $\mathrm{gr}\,\mathcal{A}$ has a Poisson bracket of degree -2 given by lifting, commuting, and projecting. $\mathrm{Fun}_{\tilde{X}}$ has a Poisson bracket of degree -2 given by ω, which has weight 2: take two functions, turn them into 1-forms, use ω to make one a vector field, then evaluate. I also want the grading on $\mathrm{gr}\,\mathcal{A}$ to match the S-grading on Fun_X (so work in conical topology).

Let $A = \Gamma(\tilde{X}, \mathcal{A})$, so A is filtered with $\mathrm{gr}\,A = \mathbb{C}[X]$.

Ex: $\tilde{X} = T^*(G/B)$ $\quad \mathcal{A} = \pi^{-1} D_{G/B}$

$$A = \mathrm{Diff}(G/B) \cong U(\mathfrak{g})_0$$

$$\begin{array}{ccc} U(\mathfrak{g}) & \twoheadrightarrow & U(\mathfrak{g})_0 \\ & & \Big\updownarrow \mathrm{gr} \\ \mathbb{C}[\mathfrak{g}] & \to & \mathbb{C}[\mathrm{nil}(\mathfrak{g})] \end{array}$$

Actually $\mathfrak{g}^*$, but we use the killing form.

Fact: $\exists!$ "extra" $\mathbb{C}^* \curvearrowright A$ so that $\mathrm{gr}\,A \cong \mathbb{C}[X]$ is compatible with both gradings

– Here's the analogue of the Beilinson-Bernstein theorem.

Thm (Braden-P-Webster): The functor $A\text{-mod} \xrightarrow{Loc} \mathcal{A}\text{-mod}$

$$N \longmapsto \mathcal{A} \otimes_A N$$

is an equivalence for "most" quantizations.

Now let's talk about support cycles.

$$\begin{array}{ccccc} A\text{-mod} & \xrightarrow{Loc} & \mathcal{A}\text{-mod} & \xrightarrow{\text{support}} & \text{cycles on } \tilde{X} \\ \cup & & & & \cup \\ \mathcal{O} & & \longrightarrow & & \text{cycles on } \tilde{X}_+ \end{array}$$

$\mathcal{O}$:= A-modules that are locally finite for the action of $A_+ \subset A$

sum of non-neg wt spaces for extra grading (analogue of $U(\mathfrak{b})$)

Prop (Braden-Licata-P-Webster): $K(\mathcal{O}) \otimes \mathbb{C} \cong H_d^{BM}(\tilde{X}_+)$

Finally, we want to use bimodules to categorify the convolution operators.

$$\begin{array}{ccccc} A\text{-bimod} & \xrightarrow{Loc} & \mathcal{A}\text{-bimod} & \xrightarrow{\text{support}} & \text{cycles on } \tilde{X} \times \tilde{X} \\ \cup & & & & \cup \\ HC & & \longrightarrow & & \text{cycles on } Z \end{array}$$

gr. supported on X_Δ

Thm (BPW): ① HC is a tensor category acting on $\mathcal{O}$
② Support intertwines $\overset{L}{\otimes}$ with $*$
③ ⇒ nice bimodules that fit together into a generalized braid group action on $D^b(\mathcal{O})$

Category $\mathcal{O}$, symplectic duality, and the Hikita conjecture

Lecture 2: The Hikita conjecture

Let me start by reminding you what a conical symplectic resolution is.

$\tilde{X}$ conical symplectic resolution

$S = \mathbb{C}^\times \curvearrowright$ $\tilde{X} \downarrow X$

- $\tilde{X}$ symplectic, symplectic form has weight 2.
- X normal, affine
- S action $\rightsquigarrow$ $\mathbb{C}[X] = \bigoplus_{n \geq 0} \mathbb{C}[X]^n$

$\mathbb{C}[X]^0 = \mathbb{C}$

$\mathbb{C}[X]^1 = 0$

This last condition is new, and it basically just rules out $X = \mathbb{C}^2$ with the scalar action (where the two coordinates x and y have weight 1, and $dx \wedge dy$ has weight 2)

$\operatorname{Aut}(X) :=$ conical symplectic automorphisms of $\tilde{X}$

$T \subset \operatorname{Aut}(X)$ maximal torus

Assume $|\tilde{X}^T| < \infty$.

Choose generic $\mathbb{C}^\times \hookrightarrow T$ "extra action"

In the first lecture I just wanted the extra $\mathbb{C}^\times$ action, but today I will want to talk about the whole extra T-action

Let's focus on two very explicit examples.

Ex I: $X = \mathbb{C}^2/(\mathbb{Z}/3\mathbb{Z})$

$= \operatorname{Spec} \mathbb{C}[x,y]^{\mathbb{Z}/3\mathbb{Z}} \qquad \zeta \cdot x = \zeta x, \quad \zeta \cdot y = \zeta^{-1} y.$

$= \operatorname{Spec} \mathbb{C}[xy, x^3, y^3]$

$= \operatorname{Spec} \mathbb{C}[a, b, c] / \langle a^3 - bc \rangle$

S scales $\mathbb{C}^2$ with weight -1 $\rightsquigarrow$ $\deg(a) = 2$, $\deg(b) = \deg(c) = 3$.

$\tilde{X} =$ 

$\downarrow$ crepant resolution

$X =$

$T = \operatorname{Aut}(X) = \mathbb{C}^\times$, $\quad t \cdot a = a$, $t \cdot b = t b$, $t \cdot c = t^{-1} c$. $\qquad \tilde{X}^T =$ 

Ex II: $X^! = \{3 \times 3$ nilpotent matrices of rank $\leq 1\}$

$\tilde{X}^! = \{(M, \ell) \mid M \in X, \ \ell \supset \operatorname{im}(M)\}$

$\uparrow$ line

$$X^! \xleftarrow{\pi_1} \tilde{X}^! \xrightarrow{\pi_2} \mathbb{P}^2$$

$\tilde{X}^! \cong T^* \mathbb{P}^2$

S scales fibers with weight -2.

$\operatorname{Aut}(X^!) = PGL_3(\mathbb{C})$

$T^! = (\mathbb{C}^\times)^3 / \mathbb{C}^\times_\Delta \quad$ 2-dim'l. $\qquad (\tilde{X}^!)^{T^!} =$ coordinate points on $\mathbb{P}^2$

Fact: X and $X^!$ are "dual" in the sense that $\mathcal{O}_X$ is koszul dual to $\mathcal{O}_{X^!}$

This kind of thing happens a lot! Here are some other examples of dual pairs.

Other examples of dual pairs:

1. $T^*(G/B)$ dual to $T^*(G^L/B^L)$ (Beilinson-Ginzburg-Soergel)
2. $\mathrm{Hilb}^n \mathbb{C}^2 / \mathbb{C}^2_\Delta$ is self-dual (Bouquier-Shan-Varagnolo-Vasserot)
3. Quiver varieties are dual to slices in Gr_G (Webster)
4. Hypertoric varieties are dual to other hypertoric varieties (BLPW)

Our concrete example is a special case of both 3 and 4.

Question: What is the coordinate ring of X^T?

X^T is a single point (there are three fixed points on $\tilde{X}$, but they all map to the same point in X), but I claim that it has a natural non-reduced scheme structure, so its coordinate ring will be a finite dimensional algebra.

Given $G \circlearrowright X = \operatorname{Spec} R$, $p \in X^G \iff f(p) = f(\sigma \cdot p) \ \forall f \in R, \sigma \in G$

$\iff f(p) = (\sigma \cdot f)(p) \ \forall f \in R, \sigma \in G$

$$\rightsquigarrow X^G = \operatorname{Spec} R / \langle \sigma \cdot f - f \mid \sigma \in G, f \in R \rangle.$$

We take this as the definition of the fixed point scheme

If $G = \mathbb{C}^*$, $\langle \sigma \cdot f - f \rangle = \langle$ all functions of weight $\neq 0 \rangle$

So $X^{\mathbb{C}^*} = \operatorname{Spec} \mathbb{C}[X]_0 / \langle fg \mid wt(f) = -wt(g) \neq 0 \rangle$

Ex: $\mathbb{C}[X] = \mathbb{C}[\overset{wt\,0}{a}, \overset{wt\,1}{b}, \overset{wt\,-1}{c}] / \langle a^3 - bc \rangle$

$$\mathbb{C}[X^T] = \mathbb{C}[a] / \langle bc \rangle$$

$$= \mathbb{C}[a] / \langle a^3 \rangle \cong H^*(T^*\mathbb{P}^2)$$

Hikita Conjecture: If X and $X^!$ are dual, then $\mathbb{C}[X^T] \cong H^*(\tilde{X}^!)$.

On the exercises, you can check that it works with X and $X^!$ reversed.

Proved by Nikita for
- Springer resolution (DiC-P)
- $\operatorname{Hilb}^n \mathbb{C}^2$
- Type A quiver varieties / Gr slices
- Hypertoric varieties

Let me record the following special case of Hikita's conjecture.

$$\boxed{\text{Degree } 2: \quad H^2(\tilde{X}^!) \underset{\substack{\uparrow \\ \text{no rels in degree 2} \\ \text{because no functions} \\ \text{of weight 1}}}{\cong} \mathbb{C}[X]^2_0 \underset{\substack{\nwarrow \\ \text{moment map}}}{\cong} \mathrm{Lie}(T)}$$

Next I want to talk about a fancier version of the Hikita conjecture in which the coordinate ring is replaced by a quantization.

Recall: - Quantization of $\tilde{X}$ = · filtered algebra A with $[A^i, A^j] \subset A^{i+j-2}$

· graded Poisson Ison $\mathrm{gr}\, A \cong \mathbb{C}[X]$

· sheafy version

Thm (Bezrukavnikov-Kaledin, Losev): Quantizations of $\tilde{X} \xrightarrow[\text{"period"}]{\sim} H^2(\tilde{X})$

Rather than explaining this bijection abstractly, let me show you how it works for our main example

First set of exercises

Ex: $\mathbb{C}[X] = \mathbb{C}[a,b,c] \big/ \langle a^3 - bc \rangle$

$$\{a,b\} = -b \qquad \{a,c\} = c \qquad \{b,c\} = 3a^2$$

$$= \mathbb{C}[a_1,a_2,a_3,b,c] \big/ \langle a_1a_2a_3 - bc,\ a_1 - a_2,\ a_2 - a_3 \rangle$$

this is obviously the same; you'll see in a moment why I want to write it this way!

let $A = \mathbb{C}\langle a_1, a_2, a_3, b, c\rangle \Big/ \left\langle \begin{matrix} [a_i, a_j] = 0 \\ [a_i, b] = b \\ [a_i, c] = -c \end{matrix} \quad \begin{matrix} bc = (a_1+1)(a_2+1)(a_3+1) \\ cb = a_1a_2a_3 \end{matrix} \right\rangle$

$$\underset{\text{center}}{Z(A)} = \mathbb{C}[\underset{x}{a_1 - a_2},\ \underset{y}{a_2 - a_3}] \cong \mathbb{C}[H^2(\tilde{X})]$$

The center of A is a polynomial ring in two variables, which we can identify with functions on $H^2(\tilde{X})$ (a 2-dim'l vector space). I claim that if we set x and y equal to numbers (there's one way to do this for each elt. of $H^2(\tilde{X})$), we get a quantization, and that each quant. arises uniquely in this way

Clear that we get a filtered alg. with $\mathrm{gr}\, A_{x,y} \cong \mathbb{C}[X]$.
How about the Poisson bracket? $\{a,b\}$ and $\{a,c\}$ are clear.

$$[b,c] = (a_1+1)(a_2+1)(a_3+1) - a_1 a_2 a_3$$

$$= a_1 a_2 + a_1 a_3 + a_2 a_3 + \text{lower}$$

$$\Rightarrow \{b,c\} = 3a^2$$

Up to now we've worked with filtered algebras, but sometimes it's convenient to use the Rees algebra construction to turn them into algebras over a polynomial ring.

Def: $A_\hbar = \mathrm{Rees}(A) = \sum \hbar^i A_i \subset \mathbb{C}[\hbar] \otimes A$

module over $\mathbb{C}[\hbar]$

$\hbar = 1 \rightsquigarrow A$

$\hbar = 0 \rightsquigarrow \mathrm{gr}\, A$ "homogenization"

Ex: $A_\hbar = \dfrac{\mathbb{C}[\hbar]\langle a_1, a_2, a_3, b, c\rangle}{\left\langle \begin{array}{ll} [a_i,a_j]=0 & bc = (a_1+\hbar)(a_2+\hbar)(a_3+\hbar) \\ [a_i,b]=-\hbar b & \\ [a_i,c]=\hbar c & cb = a_1 a_2 a_3 \end{array} \right\rangle}$

Algebra over $\mathbb{C}[H^2(\tilde{X})] \otimes \mathbb{C}[\hbar]$.

Let's regard $A_\hbar$ as a souped up version of $\mathbb{C}[X]$, and do to it the same things that we did to $\mathbb{C}[X]$

Def: $B_\hbar = (A_\hbar)_0 \Big/ \left\langle fg \;\middle|\; wt(f) = -wt(g) > 0 \right\rangle$ ← noncommutative, so now the sign matters!

Algebra over $\mathbb{C}[H^2(\tilde{X})] \otimes \mathbb{C}[\hbar]$

Why is this an interesting object to consider?

Aside. $(A_\hbar)_- \twoheadrightarrow (A_\hbar)_0 \twoheadrightarrow B.$

↑ sum of non-pos weight spaces

N B-module $\rightsquigarrow$ $V := A_\hbar \otimes_{(A_\hbar)_-} N$ $A_\hbar$-module

Specialize at $\lambda \in H^2(\tilde{X})$ and set $\hbar = 1$ to get a 'Verma module' in category $\mathcal{O}_-$

If we specialize $\hbar$ to 1 and evaluate at an element of $H^2(\tilde{X})$, we get a module over a quantization of $\tilde{X}$. By construction, A_+ acts trivially, so we get an object of $\mathcal{O}$. These objects are analogous to Verma modules (if $\tilde{X}$ is the Springer resolution, they are Verma modules), and they can be used to show that $\mathcal{O}$ is a highest weight category for most choices of period.

If you did the first set of exercises, you saw in an example that the simples all arise as

Ex: $B_\hbar = \mathbb{C}[\hbar, a_1, a_2, a_3] \big/ \langle bc \rangle$

$$= \mathbb{C}[\hbar, a_1, a_2, a_3] \big/ \langle (a_1+\hbar)(a_2+\hbar)(a_3+\hbar) \rangle$$

$$\cong H^*_{T^! \times \mathbb{C}^\times}(T^*\mathbb{P}^2)$$

algebra over $\mathbb{C}[H^2(\tilde{X})] \otimes \mathbb{C}[\hbar]$

$$\cong \mathbb{C}[\mathrm{Lie}(T)] \otimes \mathbb{C}[\hbar]$$

$$\cong H^*_{T^!}(pt) \otimes H^*_{\mathbb{C}^\times}(pt)$$

$$\cong H^*_{T^! \times \mathbb{C}^\times}(pt)$$

This motivates the equivariant version of the Hikita conjecture, which was proposed by Nakajima.

Equivariant Hikita Conj: If X and $X^!$ are dual, then $B_\hbar \cong H^*_{T^! \times \mathbb{C}^\times}(\tilde{X}^!)$ as graded algebras over $\mathbb{C}[H^2(\tilde{X})] \otimes \mathbb{C}[\hbar] \cong H^*_{T^! \times \mathbb{C}^\times}(pt)$.

Category $\mathcal{O}$, symplectic duality, and the Hikita conjecture

Lecture 3: The quantum Hikita conjecture

Let me begin by reminding you of the basic objects of study from the previous lecture.

$$\begin{array}{c} \tilde{X} \\ \downarrow \\ X \end{array} \quad \text{conical symplectic resolution}$$

Ex: $\tilde{X}$ = 

$X = \mathbb{C}^2/(\mathbb{Z}/3\mathbb{Z})$ = 

$A_\hbar$ = Rees algebra of universal quantization

Ex: $A_\hbar = \dfrac{\mathbb{C}[\hbar]\langle a_1, a_2, a_3, b, c\rangle}{\left\langle \begin{array}{ll} [a_i, a_j] = 0 & bc = (a_1+\hbar)(a_2+\hbar)(a_3+\hbar) \\ [a_i, b] = -\hbar b & cb = a_1 a_2 a_3 \\ [a_i, c] = \hbar c & \end{array} \right\rangle}$

Algebra over $\mathbb{C}[a_1 - a_2, a_2 - a_3, \hbar]$

$= \mathbb{C}[H^2(\tilde{X})] \otimes \mathbb{C}[\hbar]$.

If we set $a_1 - a_2$ and $a_2 - a_3$ equal to complex #s and set $\hbar = 1$, we get a quantization in the sense of the first lecture, and every quantization arises uniquely in this way.

$$B_\hbar = (A_\hbar)_0 \Big/ \left\langle fg \;\middle|\; wt(f) = -wt(g) > 0 \right\rangle$$

Ex: $\mathbb{C}[\hbar, a_1, a_2, a_3] / \langle bc \rangle$

$$= \mathbb{C}[\hbar, a_1, a_2, a_3] \Big/ \langle (a_1+\hbar)(a_2+\hbar)(a_3+\hbar) \rangle.$$

$$\cong H^*_{T^! \times \mathbb{C}^*}(T^*\mathbb{P}^2)$$

Eq Hikita Conjecture: If X and $X^!$ are dual, then $B_\hbar \cong H^*_{T^! \times \mathbb{C}^*}(\tilde{X}^!)$ as graded algebras over $\mathbb{C}[H^2(\tilde{X})] \otimes \mathbb{C}[\hbar] \cong H^*_{T^! \times \mathbb{C}^*}(pt)$.

Today I want to talk about a new version of this conjecture involving quantum cohomology!

$$S := \mathbb{C}[q^\lambda \mid \lambda \in H_2(\tilde{X}^!;\mathbb{Z}) \text{ effective}]$$

Ex: $\tilde{X}^! = T^*\mathbb{P}^2$

$H_2(\tilde{X}^!;\mathbb{Z}) = \mathbb{Z} > \mathbb{N}$ ← effective classes

$S = \mathbb{C}[q]$

(allow power series)

$QH^*_{T^!\times\mathbb{C}^*}(\tilde{X}^!) = H^*_{T^!\times\mathbb{C}^*}(\tilde{X}^!) \otimes \hat{S}$ with funny product

$q=0 \rightsquigarrow$ ordinary cohomology.

Thm (Braverman-Maulik-Okounkov): "In many cases", $\exists$

- finite set $\Delta_+ \subset H_2(\tilde{X}^!;\mathbb{Z})_{eff}$
- operators $L_\alpha : H_{T^!\times\mathbb{C}^*}(\tilde{X}^!) \circlearrowleft \quad \forall\ \alpha \in \Delta_+$

These are Nakajima/Heisenberg operators of the sort discussed in my first lecture.

Such that $\forall\ u \in H^2_{T^!\times\mathbb{C}^*}(\tilde{X}^!)$ and $v \in H^*_{T^!\times\mathbb{C}^*}(\tilde{X}^!)$,

$$\underset{\text{quantum}}{u * v} = \underset{\text{ordinary}}{u \cdot v} + \sum_{\alpha\in\Delta_+} \langle u,\alpha\rangle \frac{\hbar q^\alpha}{1-q^\alpha} L_\alpha(v).$$

project u to $H^2(\tilde{X}^!)$

So in fact we don't need to work over power series, we only need to invert $1-q^\alpha\ \forall\ \alpha\in\Delta_+$.

This minus sign has to do with the fact that $[a_i, f] = -\hbar\, wt(f) f$.

How does it work when $\tilde{X}^! = T^*\mathbb{P}^2$?

Ex: $\tilde{X}^! = T^*\mathbb{P}^2$

Facts: • $\Delta_+ = \{\alpha\}$ with $\langle a_i, \alpha\rangle = -1$, so $a_i * v = a_i v - \frac{\hbar q}{1-q} L(v)$

• $L(v) = 0 \quad \forall\ v \in H^0$ or H^2.

• $L(a_2 a_3) = -a_1 a_2 a_3/\hbar$

I'm going to write q instead of q^α, since there is only one α.

Why does that last statement make sense?

$(a_1+\hbar)(a_2+\hbar)(a_3+\hbar) = 0 \Rightarrow \hbar$ divides $a_1 a_2 a_3$.

$$\Rightarrow L((a_2+\hbar)(a_3+\hbar)) = L(a_2 a_3 + \hbar(a_2+a_3) + \hbar^2)$$
$$= L(a_2 a_3) + \hbar L(a_2+a_3) + \hbar^2 L(1)$$
$$= L(a_2 a_3) = -a_1 a_2 a_3/\hbar.$$

L acts by module maps

$$\Rightarrow (a_1+\hbar) * (a_2+\hbar) * (a_3+\hbar) = (a_1+\hbar) * (a_2+\hbar)(a_3+\hbar)$$
$$= (a_1+\hbar)(a_2+\hbar)(a_3+\hbar) + \frac{q}{1-q} a_1 a_2 a_3$$
$$= 0 + \frac{q}{1-q} a_1 a_2 a_3$$

divisor times divisor is unmodified

$$a_1 * a_2 * a_3 = a_1 * a_2 a_3$$
$$= a_1 a_2 a_3 + \frac{q}{1-q} a_1 a_2 a_3$$
$$= \frac{1}{1-q} a_1 a_2 a_3$$

$$\Rightarrow (a_1+\hbar) * (a_2+\hbar) * (a_3+\hbar) = q\, a_1 * a_2 * a_3$$

Prop (McBreen-Shenfeld): $H^*_{T^! \times \mathbb{C}^*}(T^*\mathbb{P}^2) = \dfrac{\mathbb{C}[a_1,a_2,a_3,\hbar][[q]]}{\langle (a_1+\hbar)(a_2+\hbar)(a_3+\hbar) - q\,a_1a_2a_3 \rangle}$

The relation we found is basically the only relation.

Okay, now what can we do with the quantized coordinate ring on the dual side to match up with the quantum cohomology?

Recall: $T \subset \mathrm{Aut}(X)$ max torus

Basic Hikita $\Rightarrow \mathrm{Lie}(T) \cong H^2(\tilde{X}^!)$

$\Rightarrow \mathrm{Hom}(T, \mathbb{C}^*) \cong H_2(\tilde{X}^!; \mathbb{Z}) > \Delta_+$

By the regular Hikita conjecture, we can regard Δ_+ as sitting inside the character lattice of T.

Def: $M := (A_\hbar)_0 \otimes S \Big/ S \cdot \left\{ fg - q^\alpha g f \;\middle|\; \begin{array}{l} wt(f) = \alpha \in \mathbb{N}\Delta_+ \\ wt(g) = -\alpha \end{array} \right\}$

Ex: In our favorite example, $wt(b) = 1$, $wt(c) = -1$, so we kill $bc - qcb = (a_1+\hbar)(a_2+\hbar)(a_3+\hbar) - q\,a_1a_2a_3$. Great!

When $q=0$, we get $B_\hbar$, which is what we want. (There's something to be checked here, but it works.) When $q=1$, we get Hochschild homology.

$$M|_{q=0} \cong B_\hbar$$
$$M|_{q=1} \cong HH_0(A_\hbar) = A_\hbar/\text{commutators}$$

Let me digress about this for a moment...

$A \curvearrowright V$ finite dimensional

$\rightsquigarrow A_\hbar \curvearrowright V_\hbar = \mathrm{Rees}(V)$

$$\rightsquigarrow \quad f \longmapsto \mathrm{tr}(f \curvearrowright V_\hbar)$$
$$\begin{tikzcd} A_\hbar \arrow[rr] \arrow[dr] & & \mathbb{C}[\hbar] \\ & HH_0(A_\hbar) \arrow[ur] & \end{tikzcd}$$

degree zero Hochschild homology is the universal source for traces of A-modules.

M provides a graded version of this construction.

$$A \curvearrowright V = \bigoplus_{\mu \in \mathrm{Hom}(T,\mathbb{C}^\times)} V_\mu \quad \text{f.d. weight spaces}$$

(eg object of $\mathcal{O}$)

$\rightsquigarrow A_\hbar \curvearrowright V_\hbar$

$$f \longmapsto \sum \mathrm{tr}(f \curvearrowright (V_\hbar)_\mu)\, q^\mu$$
$$\begin{tikzcd} (A_\hbar)_0 \arrow[rr] \arrow[dr] & & \mathbb{C}[\hbar] \otimes \mathbb{C}[q^\mu \mid \mu \in \mathrm{Hom}(T,\mathbb{C}^\times)] \\ & M \arrow[ur] & \end{tikzcd}$$

The annoying problem is that this thing that we are killing is not an ideal, and M is therefore not an algebra.

$$\boxed{\begin{aligned} & b(a_1c) - q(a_1c)b \\ = \; & bc(a_1+t) - qa_1cb \\ = \; & (a_1+t)^2(a_2+t)(a_3+t) - qa_1^2a_2a_3 \end{aligned}}$$

This is not the same as a_1 times the previous relation.

We can fix this problem by changing the multiplication!

Def: $R = \mathbb{C}[t]\langle a_1, a_2, a_3, q\rangle \Big/ \left\langle \begin{matrix} [q, a_i] = qt \\ [a_i, a_j] = 0 \end{matrix} \right\rangle$.

Then $qa_i = (a_i+t)q$

$$\Rightarrow \quad b(a_1c) - q(a_1c)b = (a_1+t)(bc - qcb)$$

Prop: M is an R-module, and

$$M \cong R \big/ R\cdot\{(a_1+t)(a_2+t)(a_3+t) - qa_1a_2a_3\}$$

How does this generalize? We have the following SES:

$$0 \to \mathbb{C}\hbar \otimes H_2(\tilde{X}) \to (A_\hbar)^2_0 \longrightarrow \mathbb{C}[X]^2_0 \longrightarrow 0$$

$$u \longmapsto \bar{u} \qquad \mathbb{C}[X]^2_0 \cong \mathrm{Lie}(T)$$

degree 2 functions on $H^2(\tilde{X})$

Now we use it to build a ring R as follows.

$$R := S \otimes \mathrm{Sym}\,(A_\hbar)^2_0$$

$$f \cdot q^\alpha = q^\alpha (u + \langle \alpha, \bar{u} \rangle \hbar)$$

$$R \curvearrowright S \otimes (A_\hbar)_0 \longrightarrow M$$

Remember that $\alpha \in \mathrm{Hom}(T, \mathbb{C}^\times)$ is a character of T, so it can be paired with a cocharacter.

What about on the dual side?

This is why I used the letter u above.

Recall: Equivariant Hikita $\Rightarrow (A_\hbar)^2_0 \cong H^2_{T^! \times \mathbb{C}^\times}(\tilde{X}^!)$.

$$R \curvearrowright QH^*_{T^! \times \mathbb{C}^\times}(\tilde{X}^!) = \hat{S} \otimes H^*_{T^! \times \mathbb{C}^\times}(\tilde{X}^!)$$

$$u \cdot (q^\alpha \otimes v) = \langle \alpha, \bar{u} \rangle q^\alpha \otimes v + q^\alpha \otimes (u * v)$$

"quantum D-module"

Quantum Hikita conjecture (Kamnitzer-McBreen-P):

$\hat{M} := \hat{S} \otimes_S M$ is isomorphic to $QH^*_{T^! \times \mathbb{C}^\times}(\tilde{X}^!)$

as a module over $\hat{R} = \hat{S} \otimes_S R$

invert $1-q^\alpha$ $\forall \alpha \in \Delta_+$.

Thm: True for Springer resolutions and hypertoric varieties.