

**SPRINGER FIBERS: BASIC PROPERTIES AND APPLICATIONS TO
CATEGORIFICATION. TALK 1**

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1. BASICS OF SPRINGER FIBERS

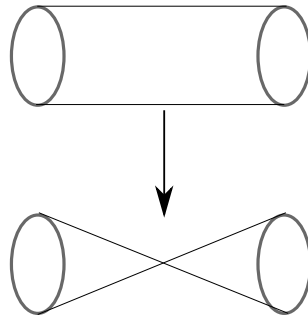
Let \mathfrak{g} be a semi-simple complex Lie algebra. Inside \mathfrak{g} we have the cone of nilpotent elements \mathcal{N} .

Example 1.1. Let $\mathfrak{g} = \mathfrak{sl}_2$. Then

$$\mathcal{N} = \left\{ \left(\begin{array}{cc} a & b \\ c & -a \end{array} \right) \in \mathfrak{sl}_2(\mathbb{C}) \mid a^2 + bc = 0 \right\}$$

The equation $a^2 + bc = 0$ ensures the matrix is nilpotent. It also defines a type A_1 singularity which is the singularity formed on the quotient $\mathbb{C}^2 / \mathbb{Z}_2$.

The variety \mathcal{N} is singular at the origin. It is resolved by $T^*\mathbb{P}^1$. We call this *the Springer resolution* in type A_1 . If we draw just the real points, we have a double cone resolved by a cylinder.



We can identify

$$\begin{aligned} T^*\mathbb{P}^1 &\leftrightarrow \{(L, A) \in \mathbb{P}^1 \times \mathcal{N} \mid A(\mathbb{C}^2) \subseteq L, A(L) \subseteq 0\} \\ (L, f) &\mapsto (L, A_f) \end{aligned}$$

as follows. A point L in \mathbb{P}^1 defines a one-dimensional subspace $L \subseteq \mathbb{C}^2$. Identifying \mathbb{C}^2 / L with $T_L \mathbb{P}^1$ we use $f : T_L \mathbb{P}^1 \rightarrow \mathbb{C}$ to define A_f via $A_f(y) = f(y)L$ and $A_f(L) = 0$. With respect to this identification the Springer resolution is

$$\mu : (L, A) \mapsto A.$$

In general, we identify

$$T^*(G/B) = \{(b, x) \mid b \in \mathfrak{g} \text{ borel}, x \in \mathfrak{b} \text{ nilpotent}\}$$

and then define *the Springer resolution* as

$$\begin{aligned} \mu : T^*(G/B) &\longrightarrow \mathcal{N} \\ (\mathfrak{b}, x) &\longmapsto x. \end{aligned}$$

In type A as in the example for \mathfrak{sl}_2 , we can describe $T^*(G/B)$ in terms of flags

$$T^*(SL_n/B) = \{(F_\bullet, x) \mid 0 = F_0 \subset F_1 \subset \dots \subset F_n = \mathbb{C}^n, \dim(F_i/F_{i-1}) = 1, x(F_i) \subset F_{i-1}\}$$

Remark 1.2. When $\mathfrak{g} = \mathfrak{so}$ or \mathfrak{sp} we can give similar descriptions of $T^*(G/B)$ in terms of flags. In this case \mathbb{C}^m for $m = 2n$ or $2n + 1$ has a bilinear form and we use flags F_\bullet such that $F_i = F_{m-i}^\perp$.

Definition 1.3. The Springer fiber is

$$\mathcal{B}_x = \mu^{-1}(x).$$

Note that the Springer fiber only depends on which G -orbit x is in. Thus, in type A , for example, \mathcal{B}_x is specified by a Jordan type.

Example 1.4. Let $\mathfrak{g} = \mathfrak{sl}_n$. Let $x = 0$. Then $\mathcal{B}_x = G/B$.

Example 1.5. Let x be a regular nilpotent element

$$\begin{pmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & \\ \vdots & & & \ddots \end{pmatrix},$$

then $\mathcal{B}_x = \{\text{pt}\}$.

Example 1.6. Let $\mathfrak{g} = \mathfrak{sl}_3$. Choose $\{e_1, e_2, f\}$ a basis. Then let x be

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To compute \mathcal{B}_x we need to find flags $F_1 \subset F_2 \subset \mathbb{C}^3$ stabilized by x . Two such families are

$$\begin{aligned} C_1^\circ &= \{\langle e_1 \rangle \subset \langle e_1, e_2 + bf \rangle \subset \mathbb{C}^3 \mid b \in \mathbb{C}\}, \\ C_2^\circ &= \{\langle e_1 + bf \rangle \subset \langle e_1 + bf, e_2 \rangle \subset \mathbb{C}^3 \mid b \in \mathbb{C}\}. \end{aligned}$$

The closures of these are

$$\begin{aligned} C_1 &= \overline{C_1^\circ} = \{\langle e_1 \rangle \subset \langle e_1, Ae_2 + Bf \rangle \subset \mathbb{C}^3 \mid [A : B] \in \mathbb{P}^1\}, \\ C_2 &= \overline{C_2^\circ} = \{\langle Ae_1 + Bf \rangle \subset \langle Ae_1 + Bf, e_2 \rangle \subset \mathbb{C}^3 \mid [A : B] \in \mathbb{P}^1\}. \end{aligned}$$

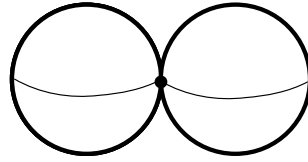
These are each isomorphic to \mathbb{P}^1 . Their intersection

$$C_1 \cap C_2 = \{\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \mathbb{C}^3 \mid [A : B] \in \mathbb{P}^1\}$$

is just one point. We *claim* that the union is $C_1 \cup C_2 = \mathcal{B}_x$. If we assume the claim, then we have shown

$$\mathcal{B} = \mathbb{P}^1 \bigwedge \mathbb{P}^1.$$

as illustrated below.



This is an example of an A_2 singularity.

Remark 1.7. The action of the torus T on G/B does not descend to \mathcal{B}_x . Only those elements in

$$C_T(x) = \{t \in T \mid tx = xt\}$$

act on \mathcal{B}_x .

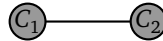
Example 1.8. In [Example 1.6](#), we have

$$C_T(x) = \left\{ \left(\begin{array}{ccc} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-1} \end{array} \right) \mid t \in \mathbb{C}^* \right\}$$

In type A , if x has Jordan type λ , i.e. λ is a partition of n , then the $C_T(x)$ -fixed points of \mathcal{B} are in one-to-one correspondence with S_n/S_λ . This correspondence is given by considering the flags generated by adding coordinate vectors $\{e_1, \dots, e_n\}$ one at a time which are preserved by x .

Definition 1.9. We define the *intersection graph* of \mathcal{B}_x to have a vertex for each irreducible component C_i and an edge between C_i and C_j if their intersection is non-empty. The *advanced intersection graph* has the same vertex set, but is directed with $m_{ij} = \dim(H^*(C_i \cap C_j))$ arrows from C_i to C_j .

Example 1.10. In [Example 1.6](#), the intersection graph is



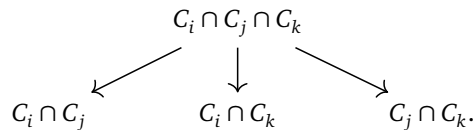
The advanced intersection graph is



By Spaltenstein, the irreducible components of \mathcal{B}_x all have the same dimension. We can define a *convolution algebra* structure on

$$A_{\text{conv}} = \bigoplus_{(C_i, C_j)} H^*(C_i \cap C_j)$$

using the usual push-pull convolution product on



This behaves well if the intersections $C_i \cap C_j$ are smooth.

Special Cases:

- (1) If $x = 0$ then $A_{\text{conv}} = H^*(G/B)$ leads to Soergel bimodules.
- (2) In type A with x of Jordan type $(n - 1, 1)$, the algebra A_{conv} is the Khovanov-Seidel algebra. (It arises from the Fukaya category of the Milnor fiber.)

- (3) The principal block of $\mathcal{U}_q(\mathfrak{sl}_2)$ when $q \neq \pm 1$ is a root of unity (Anderson-Tubbenhauer, Arkhipov-Bezrukavnikov-Ginzburg).
- (4) The principal block of \mathcal{O} of finite dimensional representations of $\mathfrak{gl}(1|1)$.

Seven Ideas From an Old Theorem

Theorem 1.11 (S.-Webster). *Let $x \in \mathfrak{sl}_n$ with 1 or 2 Jordan blocks. Then:*

- (1) *Let Z be a nilpotent slice and $i : \mathcal{B}_x \hookrightarrow Z$. Then*

$$A_{\text{conv}} \cong \bigoplus_{(C_1, C_2)} \text{Ext}_{\text{Coh}(Z)}^\bullet \left(i_* \Omega_{C_1}^{1/2}, i_* \Omega_{C_2}^{1/2} \right).$$

- (2) *Let \mathfrak{p}_x be the parabolic corresponding to the Jordan type of x . Let $\mathcal{O}_0^{\mathfrak{p}_x}(\mathfrak{sl}_n)$ be the principal block of parabolic category \mathcal{O} for \mathfrak{sl}_n . Then there exists a fully faithful functor*

$$\mathcal{O}_0^{\mathfrak{p}_x}(\mathfrak{sl}_n) \longrightarrow A_{\text{conv}}.$$

- (3) *Let p be a fixed point of the $C_T(x)$ action on \mathcal{B}_x . Define C_p to be the closure of the attracting cell in \mathcal{B}_x to p . If we fix a cocharacter $\mathbb{C}^* \hookrightarrow T$, then we can write*

$$C_p = \{x \mid \lim_{t \rightarrow 0} tx = p\}$$

Summing over pairs of such fixed points define

$$A'_{\text{conv}} = \bigoplus_{(p, p')} H^*(C_p \cap C_{p'}).$$

Then

$$\mathcal{O}_0^{\mathfrak{p}_x}(\mathfrak{sl}_n) \cong A'_{\text{conv}}.$$

Seven Ideas:

Idea 1. Think of A_{conv} as an algebraic version of the Fukaya category.

Idea 2. The $i_* \Omega_{C_i}^{1/2}$ are simple objects in an “exotic” t -structure.

Idea 3. Think of

$$\text{KZ} : \mathcal{O}_0^{\mathfrak{p}_x}(\mathfrak{sl}_n) \longrightarrow A_{\text{conv}}\text{-mod}.$$

Idea 4. Use this to give a geometric construction of category \mathcal{O} .

Idea 5. Consider $\mathcal{O}_0^{\mathfrak{p}_x}(\mathfrak{sl}_n)$ as a wrapped Fukaya category.

Idea 6. The functor

$$A_{\text{conv}}\text{-mod} \longrightarrow A'_{\text{conv}}\text{-mod}$$

is *Schurification* (in the sense of Roquier).

Idea 7. There is a homological grading on

$$H^*(C_p \cap C_{p'}).$$

Classification of Irreducible Components of \mathcal{B}_x in Types ABCD.

In [Example 1.6](#), there was a Jordan sequence for each component as such

$$C_1 \begin{array}{|c|} \hline 1 & 3 \\ \hline 2 \\ \hline \end{array} \longrightarrow \begin{array}{|c|} \hline \\ \hline \end{array} \longrightarrow \square$$

$$C_2 \begin{array}{|c|} \hline 1 & 2 \\ \hline 3 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|} \hline & \\ \hline \end{array} \longrightarrow \square$$

given by starting with the partition corresponding to the nilpotent orbit of x and removing a box at each step corresponding to how x interacts with the F_i .

Theorem 1.12 (Vargos-Spaltenstein-Springer, Van Leeuwen). *Let λ be the partition corresponding to a nilpotent orbit. Then there is a one-to-one correspondence (which in type A is)*

$$\begin{aligned} \{\text{components of } \mathcal{B}_x\} &\leftrightarrow \{\text{standard tableau of shape } \lambda\} \\ \{F_\bullet \mid F_\bullet \text{ has Jordan sequence } T\} &\leftrightarrow T. \end{aligned}$$

In types BCD, we would need to use signed domino tableau, but a similar correspondence holds.

We can describe the components very well in two special cases.

[Ehrig-S.] For λ in type ABCD with 2 rows, we have an explicit description of all components. Note that pairwise intersections of components are always iterated \mathbb{P}^1 -bundles. The diagramatics in this case leads to Khovanov algebras.

[Sarbin] For λ a hook partition, we get iterated partial flag variety bundles.

[Fresse-Melnikov] The partition λ is 2-row or hook in type A if and only if all the components are smooth.

Final Observations.

By Springer, there is an action of W , the Weyl group, on $H^*(\mathcal{B}_x)$.

Lusztig showed in type A,

$$H^*(\mathcal{B}_\lambda) \cong \text{Ind}_{S_\lambda}^{S_n}(\text{triv}).$$

By Schur-Weyl duality, if V is the natural \mathfrak{sl}_2 -module, $V^{\otimes d}$ carries actions of \mathfrak{sl}_2 and S_d . The representations which occur are labeled by 2-row partitions. The weight spaces are permutation modules. When we quantize, we get the Jones polynomial from knot theory.

Similarly if V is the natural representation of $\mathfrak{gl}(1|1)$, then $V^{\otimes d}$ carries actions of $\mathfrak{gl}(1|1)$ and S_d . The representations which occur are labeled by hook partitions. Quantization gives the Alexander polynomial from knot theory.