10/07/2018

(Stroppel II)

Springer-Lusztig theory

Springer: $H^*(\mathcal{B}_x) \circlearrowleft W$ ← Weyl group

Rk. The action doesn't come from an action on $\mathcal{B}_x$

Type A: $H^{top}(\mathcal{B}_\lambda) = \text{span}\{\text{irred components}\} \cong \text{Specht}(\lambda^t)$ (λ: partition)

furthermore • it is an irred. rep.

• works over $\mathbb{Z}$ (⇝ this suggests there could be a categorification)

Outside type A: additionally component group

$$H^t(\mathcal{B}_x) \text{ irred for } C_G(x)/C^\circ_G(x) \times W$$

Thm (Lusztig, Garcia-Procesi, Tanisaki)

$\mathcal{B}_x \hookrightarrow G/B \Rightarrow H^*(G/B) \twoheadrightarrow H^*(\mathcal{B}_x)$ surjective in type A.

In general false

Question Can we construct Springer rep.? "in modern way"

Basic idea: $\mathcal{A}$ abelian cat., $F: \mathcal{A} \to \mathcal{A}$ autoequiv.

Then ∃ 2 isomorphisms:

$$\begin{aligned} Z(\mathcal{A}) &\longrightarrow \mathrm{End}(F) \\ z &\longmapsto F(z) \\ z &\longmapsto z_F \end{aligned}$$

⇝ we get a linear endo. φ_F, $\varphi_F: Z(\mathcal{A}) \to Z(\mathcal{A})$ s.t. $F(z) = \varphi_F(z)_F$

[Recall $Z(\mathcal{A}) := \mathrm{End}(\mathrm{id})$, center of $\mathcal{A}$

Ex: $Z(A\text{-mod}) \cong Z(A)$]

Assume $G = \langle g_i \mid i \in I \rangle / \text{rel.}$ acts on $\mathcal{A}$; then get induced action on $Z(\mathcal{A})$

Problem: very unlikely!!

But: Braid group action on <u>derived</u> cat. are "everywhere":

- via spherical objects (Seidel-Thomas).
- $\boxed{\text{tilting complexes (Rickard)}}$

<u>Theorem</u> (Rickard)

A k-algebra, $T \in \mathcal{D}(A)$ complex of A-modules

- T perfect
- T generates $\mathcal{D}(A)$ (as triang. cat. with infinite direct sums)
- $\operatorname{Hom}_{\mathcal{D}(A)}(T, T[m]) = \begin{cases} 0 & m \neq 0 \\ B & m = 0 \end{cases}$

($\rightsquigarrow$ T <u>tilting complex</u>)

Then: $\mathcal{D}^{(b)}(A) \xrightarrow{\sim} \mathcal{D}^{(b)}(B)$ (given by some (B,A)-bimodule)

<u>Ex.</u> (M1) Tilting <u>module</u> (Happel)

(gldim $T < \infty$, $\operatorname{Ext}^i(T,T) = 0$ $i > 0$;
$A \hookrightarrow T_0 \to T_1 \to \dots \to T_n \to 0$
coresolution $T_i \in \operatorname{Add}(T)$)

(M2) (Full) tilting module $T = \bigoplus_\lambda T(\lambda)$ for $T(\lambda)$ tilting module for a quasi-hereditary algebra.

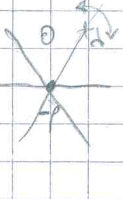
<u>Ex.</u> 0) $A = A'_{conv}$ for $(n-1,1)$-nilpotent

e_i primitive idemp. $(1 \leq i \leq n-1)$ $T_i := (Ae_i \otimes e_iA \xrightarrow{\text{"mult"}} A)$ tilting complex

(in this case, $B = A$)

1) $\mathfrak{g}$ s.s. Lie alg., $s \in W$ simple reflection
$\cup$
$\mathfrak{h}$ Cartan $S(\mathfrak{h}) =: S$

Soergel bimodules: $S \otimes_{S^s} S \xrightarrow[\text{adj}]{\text{mult.}} S$ tilting complex in $K^b(\text{Soergel-bimods}) \subseteq K^b(S\text{-gmod-}S)$

2) $\mathcal{O}_0(\mathfrak{g})$  $s \in W$ simple reflection.

then we have a couple of adjoint functors

$\Theta_{on} : pr_\lambda(- \otimes L(\lambda))\, incl_0 : \mathcal{O}_0 \to \mathcal{O}_\lambda$ transl. to the wall

$\Theta_{out} : pr_0(- \otimes L(\lambda)^*)\, incl_\lambda : \mathcal{O}_\lambda \to \mathcal{O}_0$ transl. "out off" the wall

$\rightsquigarrow \Theta_s := \Theta_{out}\Theta_{on}$ and then we have

$Id \xrightarrow{adj} \Theta_s$ "shuffling"

$\Theta_s \xrightarrow{adj} Id$ "coshuffling"

} tilting cpx.

3) $D^b_c(G/B)$ similarly $\pi_x : G/B \to G/P_x$

4) ${}^\infty_0\mathcal{H}^\infty_0$ Harish-Chandra bimodules

$\cup$

${}^\infty_0\mathcal{H}^1_0 \cong {}^1_\infty\mathcal{H}^\infty_\infty$ functors as in 2) from both sides

$\cong \qquad\qquad \cong$ (Co)twisting resp. (Co)shuffling

$\mathcal{O}_0 \qquad\quad \mathcal{O}_0$Nick's (Nick's version of $\mathcal{O}_0$)

related by Koszul-duality (Mazorchuk-Stroppel)

<u>Theorem</u> (Rouquier, Mazorchuk-Stroppel)

They all give rise to braid group actions.

<u>Rk</u> Khovanov-Seidel: in 0) have a <u>faithful</u> action

$\Longrightarrow$ <u>all</u> these actions are faithful

(A_{conv}-mod $= \mathcal{O}_0^{P_x}(sl_n)$, exercise from yesterday)

<u>Rk.</u> Passing to K_0: action of $\mathbb{C}[W]$ or Hecke algebra (if we include grading)

<u>Rk.</u> Khovanov-Mazorchuk-Stroppel

$K_0(\mathcal{O}_0^{P_x}(sl_n)) \circlearrowleft W$ <u>categorifies</u> the Springer repr. for $\mathcal{B}_x$.

In particular, it gives a <u>canonical</u> basis on $H^{top}(\mathcal{B}_x)$ by classes of indec. proj-inj. objects. Moreover the inner product on $K_0(\mathcal{O}_0^{P_x}(sl_n))$ induced by the Ext-form corresponds to the usual inner product on the Specht module $\text{Specht}(\lambda^t)$.

Rickard's theory $\Rightarrow$ action of braid group $Br(W)$ on the center $Z(\mathcal{A})$ in case of $\mathcal{D}(\mathcal{A})$.

underlying abelian cat.!!

<u>Theorem</u> Let x be a nilpotent which is regular nilp. in some Levi l_x with corresponding parabolic p_x. Then there is a commutative diagram of W-modules

$$\begin{tikzcd}
 & & Z(U(\mathfrak{g}^\vee)) \arrow[dl, "\text{H-C homo.}"'] \arrow[ddr, "\text{acting}"] & \\
 & S(\mathfrak{h}^\vee) \arrow[dl, "\text{Borel construction}"'] & & \\
H^*(G/B) \arrow[rrr, "\sim"] \arrow[d, "can"'] & & & Z(\mathcal{O}_0(\mathfrak{g})) \arrow[d, "res"] \\
H^*(\mathcal{B}_x) \arrow[rrr, dashed, "\Phi"] & & & Z(\mathcal{O}_0^{p_x}(\mathfrak{g}))
\end{tikzcd}$$

Moreover:
- Φ is an isomorphism in type A (of rings with W-action) (Brundan, Stroppel independently)
- iso. in type D <u>if</u> can is surjective
- iso. in 2-row case

Conjecture: Φ iso. whenever component group trivial

<u>Rks</u> - the "top part of the diagram" is ok as algebras: easy consequence of Soergel's Endomorphismensatz & Struktursatz (Stroppel "TQFT with corners")

- Existence of Φ: uses deformation theory and description of $H^*_{C(x)}(\mathcal{B}_x) \cong \mathbb{C}[Z_x]$, where $Z_x = \{(w(y), y) \mid w \in W,\ y \in \mathfrak{h}\} \subset \mathfrak{h} \oplus \mathfrak{h}$

$\mathfrak{h} = \mathfrak{h}_x \oplus \mathfrak{h}^x$ ← $\bigcap_{\alpha \in W_x} \operatorname{Ker} \alpha$

Cartan of Levi

- Injectivity: $H^{top}(\mathcal{B}_x)$ is sole, irred. W-module
- Surjectivity: uses outside knowledge:

$$\left\{ \begin{array}{l} H_d \curvearrowright M \otimes V^{\otimes d} \\ \qquad \mathfrak{sl}_n\text{-module} \\ \text{main idea: } "Z(H_d^{cycl}) \underset{\text{via categorification techniques}}{\cong} Z(\mathcal{O}_0^p)" \end{array} \right.$$

More details about that in Lecture III.