12/07

Catharina
Lecture 3.

(Degenerate) affine Hecke algebras.

Theorem: (Drinfeld, Ar. - Suzuki)

M $\mathfrak{gl}_n$-module, $V = \mathbb{C}^n$ nat. rep.

$\mathfrak{gl}_n \circlearrowright M \otimes V^{\otimes d} \circlearrowleft \boxed{H_d}$ deg. affine Hecke

commuting action

gen. by $\mathbb{C}[S_d]$, $\mathbb{C}[x_1, \dots, x_d]$

+ relation $s_i x_j = x_j s_i \quad j \notin \{i, i+1\}$

$s_i x_i = x_{i+1} s_i - 1$

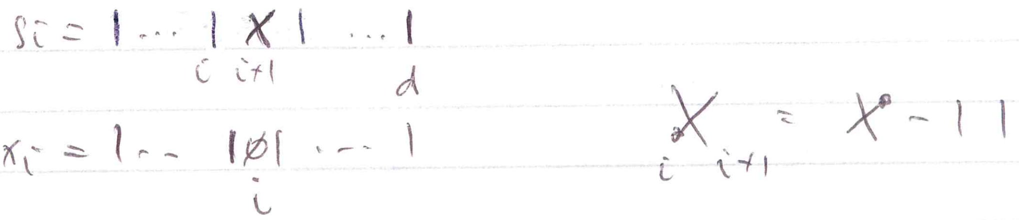
x_1 acts by $\Omega = \sum E_{ij} \otimes E_{ji}$ on $M \otimes V$

Rmk: $\mathrm{gr}\, H_d \simeq \mathbb{C}[S_d] \overset{\text{semi-direct product}}{\#} \mathbb{C}[x_1, \dots, x_d]$

$$Z(\mathrm{gr}\, H_d) = \mathbb{C}[x_1, \dots, x_d]^{S_d}$$
$$\rotatebox{90}{$\subseteq$}$$
$$Z(H_d)$$

Fact: $\mathrm{gr}\, Z(A) \subseteq Z(\mathrm{gr}\, A)$ $\quad A$ filtered

diagrams:



Theorem: (Daugharty-Ram, Eting-S)

$\mathfrak{g}$ type B, C, D, V natural repr.

M any hw. $\mathfrak{g}$-module

$\mathfrak{g} \circlearrowright M \otimes V^{\otimes d} \circlearrowleft W_d$ (par. dep.) on $\mathfrak{g}, M$

W_d gen by $Br_d(\overset{\dim V}{par})$, $\mathbb{C}[x_1, \dots, x_d]$

$\uparrow$ Brauer alg.

x_1 acts by $\sum x_i \otimes x^i$

x_i Basis of $\mathfrak{g}$

Rmk: quantized

H_d^{aff} affine Hecke $\overset{deg.}{\rightsquigarrow} H_d$

BMW $\rightsquigarrow \mathbb{W}_d$

Br_d (par) like $\mathbb{C}[S_d]$ with add. generators

$e_i = | \cdots | \substack{\cup \\ \cap} | \cdots |$ $\quad v \otimes w \mapsto (v,w) \sum v_i \otimes v^i$

$1 \le i \le d-1 \qquad i \;\; i+1 \qquad\qquad \uparrow$

Basis of V

rel. eg: $s_i x_i = x_{i+1} s_i - 1 - e_i$

parameters ☺

Theorem: (Brundan-K., Brundan-S)

type A: let $\mathfrak{p} \subseteq \mathfrak{gl}_n$ be a standard parabolic

Then 1) $\exists$ choice $M = M^{\mathfrak{p}}(\lambda)$ simple, proj. such that the actions centralise each other, in particular

$$H_d \twoheadrightarrow \mathrm{End}_{\mathfrak{g}}(M^{\mathfrak{p}}(\lambda) \otimes V^{\otimes d})$$

2) This factors through

H_d / I_ℓ, where $I_\ell = \left(\prod_{i=1}^{\ell} (x_1 - a_i) \right)$ $\uparrow$ give the size

ℓ = # blocks in $\mathfrak{p}$

3) $\exists$ central idempotent e s.t.

$$H_d / I_\ell \, e \xrightarrow{\sim} E$$

Morally: understand E = understanding $\mathcal{O}^{\mathfrak{p}}$

$\uparrow$ controlled by $\mathcal{O}^{\mathfrak{p}}$

Rmk: $\mathrm{gr}({}^{H_d}/_{I_e}) \cong \mathbb{C}[S_d] \# \mathbb{C}[x_1,\dots,x_d]/(x_i^e)$

$\exists$ Explicit bases for $Z(\mathrm{gr}\, H_d/I_e)$ by "coloured cycles"

depends on the position in the l-part. → $c^{(r)} = h_r(x_{i_1},\dots,x_{i_n}) <$ indexed by l-multipartitions of d.

complete symmetric ↗ h_r; numbers in the cycle ↗ $x_{i_1},\dots,x_{i_n}$; C-cycle ↑ $<$

Gives a description of $Z(H_d/I_e)$ $\overset{\text{Brandon}}{\rightsquigarrow}$ $Z(\mathcal{U}(\mathfrak{g}))$

$$\begin{tikzcd} Z(\mathcal{U}(\mathfrak{g})) \arrow[d, two heads] \\ Z({}^{H_d}/_{I_e}) \end{tikzcd}$$

$\longrightarrow Z(\mathcal{U}(\mathfrak{g})) \twoheadrightarrow Z(\mathcal{O}_0^p(\mathfrak{g}))$

→ surjectivity from last lecture.

Rmk: Etingof- S., S. Analogous results for BCD.

Behind this: $\mathcal{U}(\mathfrak{gl}_\infty^+)$-action on category $\mathcal{O}$.

$E_i = pr_i(\_ \otimes V)$ pr_i = projection onto ith generalised eigenspace.

from the $\mathbb{C}[S_d]$-action.

KLR:
- Put a grading on H_d
- understand interaction with eigenspaces.

Quiver Hecke algebra (KLR-algebras)

Now H_d^{aff}. Consider a quiver Q (say A_∞ or $\tilde{A}_\ell$ (cyclic))

$$\begin{tikzcd}
F_1(1) \subseteq F_1(2) \subseteq & \overset{1}{\mathbb{C}^{d_1}} \arrow[r, "f_1"] & \overset{2}{\mathbb{C}^{d_2}} \arrow[d, "f_2"] \\
F_1(e) \subseteq F_2(e) \subseteq & \overset{e}{\mathbb{C}^{d_e}} \arrow[u] & \overset{3}{\mathbb{C}^{d_3}} \arrow[l, "f_3"]
\end{tikzcd}$$

Fix $\underline{d}$ dim vector $\rightsquigarrow \mathrm{Rep}_{\underline{d}}$.

$$Q = \{ (\underset{\overset{\uparrow}{\mathrm{Rep}_{\underline{d}}}}{X}, \mathcal{F}) \mid \mathcal{F} \text{ filtration of } X \text{ s.t. the } gr_{\mathcal{F}} X \text{ ss} \}$$

flagged rep

$$\begin{tikzcd}
& \mathrm{Rep}_{\underline{d}} \\
Q(\check{\lambda}) = \{(X, \mathcal{F}) \text{ of type } \check{\lambda}\}. \arrow[ur] \arrow[d] & \\
GL_{\underline{d}}/P_{\check{\lambda}} &
\end{tikzcd}$$

$\check{\lambda}$ is a multi-dimension vector

"Steinberg variety"

$$Z(\hat{\lambda}, \hat{\mu}) = Q(\check{\lambda}) \times_{\mathrm{Rep}_{\underline{d}}} Q(\hat{\mu})$$

Defn: Quiver-Schur algebra (with conv. product)

$$A_{\underline{d}} = \bigoplus_{(\check{\lambda}, \check{\mu})} H^{BM}_{GL_{\underline{d}}}(Z(\check{\lambda}, \check{\mu}))$$

Quiver-Hecke

$$R_{\underline{d}} = \bigoplus_{(\lambda, \hat{\mu})} H \ldots \quad \leftarrow \text{entries are} \leq 1$$

Fact: $Z(H_\lambda^{aff}) = \mathbb{C}[x_1^{\pm}, \ldots, x_d^{\pm}]^{S_d}$

Define affine Schur alg. $\quad \mathrm{End}_{H_\lambda^{aff}}(\bigoplus \mathbb{C}[\underset{\overset{\uparrow}{\text{Iwahori}}}{I} \backslash G / \underset{\overset{\uparrow}{\text{Parab}}}{P}])$

$G = GL_n(\mathbb{Q}_p)$

Pick maximal ideal in $Z_d(H_d^{aff})$ corr. to the point $(q^{a_1}, \dots, q^{a_d})$, $1 \le a_i \le e$, q is e^{th} root of unity

Thm (Rouquier, VV, Miemietz - S.)

There are isomorphisms of algebras

$\hat{H}_{\underline{a}}^{aff} \simeq \hat{R}_{\underline{d}}$ $\quad \underline{d}$ dim vector corr to $\underline{a}$

completion at $m_{\underline{a}}$

$\hat{S}_{\underline{a}}^{aff} \simeq \hat{A}_{\underline{d}}$ ⇝ $\boxed{\text{gradings !}}$

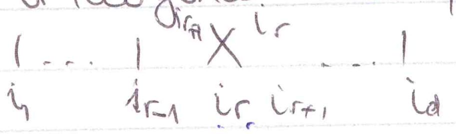
generators:

$e(\underline{i}) =$ 

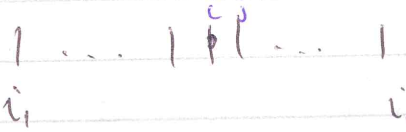
$1 \le i_j \le e$

deg 0

pairwise orthogonal idempotents

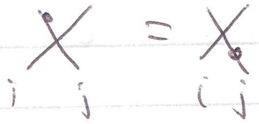
$\psi_r(i) =$ 

$\begin{cases} 0 & \text{other} \\ -2 & i=j \\ 1 & i-j \text{ in } Q \end{cases}$

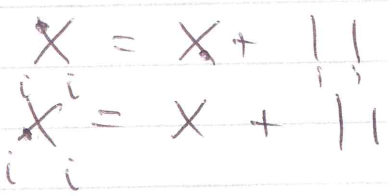
$y_r(\underline{i}) =$ 

deg 2

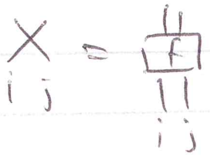
relations:



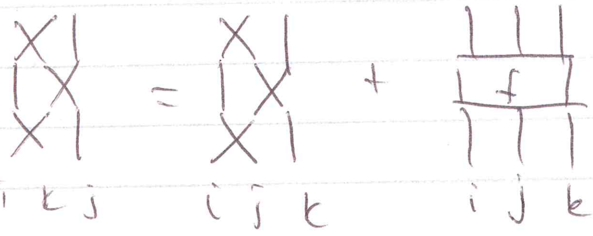
if $i \ne j$



0 $i=j$



$i \ne j$



if $i=j \ne k$